Gauss and Codazzi equations in a subspace of a Kähler space

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1. *Preliminaries*

In this paper we shall obtain Gauss and Codazzi equations in a subspace of a Kähler space. Let us consider a Kähler space K_{2n} covered by a system of complex co-ordinate neighbourhoods.

(1. 1) $$z^{\varkappa} = \xi^{\varkappa} + i\xi^{\bar{\varkappa}} \quad {}^{*)}$$

and a subspace K_{2m} $(m<n)$ defined by

(1. 2) $$z^{\varkappa} = \xi^{\varkappa}(\eta^{a}) + i\xi^{\bar{\varkappa}}(\eta^{a}) \quad {}^{**)}$$

where η^a are parameters on the subspace.

Introduce a complex co-ordinate system (u^{α}) defined by

(1. 3) $$u^{\alpha} = \eta^{\alpha} + i\eta^{\bar{\alpha}}$$

Representing both the Kähler space K_{2n} and the analytic subspace K_{2m} by the real co-ordinates ξ^h and η^a respectively, we have

(1. 4) $$\xi^h = \xi^h(\eta^a)$$

The tensor $'F_b{}^a$ is such that

(1. 5) $$F_i{}^h B_b{}^i = {}'F_b{}^a B_a{}^h$$

where $B_b{}^h = \partial_b \xi^h$

and

(1. 6) $${}'F_c{}^b\ {}'F_b{}^a = -A_c{}^a$$

The induced fundamental metric $'g_{ab}$ is given by

(1. 7) $${}'g_{cb} = B_c{}^j B_b{}^{i\,\prime} g_{ji}$$

and

(1. 8) $${}'g_{cb} = {}'F_c{}^f\, {}'F_b{}^e\, {}'g_{fe}$$

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*) The notations used in this paper are from Yano's book

**) The indices $a, b, c, \ldots$ take values from $1, 2, \ldots, m\ \bar{1}, \bar{2}, \ldots, \bar{m}$, and $\alpha, \beta, \ldots = 1, 2, \ldots, m$.

The Waerden—Bortolotti derivative of B_b^h is given by ([1], p. 103)

$$'\nabla_c B_b^h = \partial_c B_b^h + B_c^j B_b^i \begin{Bmatrix} h \\ j\ i \end{Bmatrix} - B_a^h \begin{Bmatrix} a \\ c\ b \end{Bmatrix}' \tag{1.9}$$

which is a vector of K_{2n}, orthogonal to K_{2m}.

Let us denote $2n-2m$ mutually orthogonal unit normals to K_{2m} by C_x^h, where

$$x, y, z = m+1, \dots n, \overline{m+1}, \dots, \bar{n}.$$

Thus ([1], p. 108)

$$'\nabla_c B_b^h = H_{cbx} C_x^h \tag{1.10}$$

where H_{cbx} are the second fundamental tensors of K_{2m} with respect to the normals C_x^h. C_x^h satisfies the following relations

$$B_c^j C_x^i g_{ji} = 0, \tag{1.11a}$$

$$C_y^j C_x^i g_{ji} \delta_{yx}. \tag{1.11b}$$

The equation of Weingarten are given by ([1], p. 106)

$$'\nabla_c C_x^h = -H_{cx}{}^a B_a^h + L_{cxy} C_y^h, \tag{1.12}$$

where

$$L_{cxy} = ('\nabla_c C_x^j) C_y^i g_{ji}.$$

2. *Gauss and Codazzi Equations*

Taking Waerden—Bortolotti derivative of (1. 9) we get

$$\begin{aligned} '\nabla_d '\nabla_c B_b^h &= '\nabla_d(\partial_c B_b^h) + '\nabla_d \left(B_c^j B_b^i \begin{Bmatrix} h \\ j\ i \end{Bmatrix} \right) - '\nabla_d B_a^h \begin{Bmatrix} a \\ c\ b \end{Bmatrix}' = \\ &= \partial_d \partial_c B_b^h + \partial_c B_b^l B_d^f \begin{Bmatrix} h \\ f\ l \end{Bmatrix} - \partial_l B_b^h \begin{Bmatrix} l \\ d\ c \end{Bmatrix}' - \partial_c B_l^h \begin{Bmatrix} l \\ d\ b \end{Bmatrix}' + \\ &+ \left[B_c^j (\partial_d B_b^i) \begin{Bmatrix} h \\ j\ i \end{Bmatrix} + B_b^i (\partial_d B_c^j) \begin{Bmatrix} h \\ i\ j \end{Bmatrix} + B_c^j B_b^i \partial_d \begin{Bmatrix} h \\ j\ i \end{Bmatrix} + \right. \\ &+ B_c^j B_b^i \begin{Bmatrix} l \\ j\ i \end{Bmatrix} \begin{Bmatrix} h \\ l\ d \end{Bmatrix} - B_l^j B_b^i \begin{Bmatrix} h \\ i\ j \end{Bmatrix}' \begin{Bmatrix} l \\ c\ d \end{Bmatrix} - \left. B_c^j B_l^i \begin{Bmatrix} h \\ j\ i \end{Bmatrix}' \begin{Bmatrix} l \\ b\ d \end{Bmatrix} \right] - \\ &- \left[\partial_d B_a^h \begin{Bmatrix} a \\ c\ b \end{Bmatrix}' + \partial_d \begin{Bmatrix} a \\ c\ b \end{Bmatrix}' B_a^h + B_a^l B_d^f \begin{Bmatrix} h \\ f\ l \end{Bmatrix}' \begin{Bmatrix} a \\ c\ b \end{Bmatrix} - \right. \\ &- \left. B_a^h \begin{Bmatrix} a \\ l\ b \end{Bmatrix}' \begin{Bmatrix} l \\ d\ c \end{Bmatrix}' - B_a^h \begin{Bmatrix} a \\ c\ l \end{Bmatrix}' \begin{Bmatrix} l \\ b\ d \end{Bmatrix} \right]. \end{aligned} \tag{2.1}$$

In (2. 1), on changing the indices c, d and on substracting it from the new equation, we obtain the following on simplification

$$'\nabla_d '\nabla_c B_b^h - '\nabla_c '\nabla_d B_b^h = B_a^h {}' R^a{}_{dcb} + B_b^i B_c^j K_{jdi}{}^h - B_b^i S_{cjd}{}^h, \tag{2.2}$$

where we have put

$$'R^a{}_{dcb} = \partial_c \left\{ {a \atop d\ b} \right\}' - \partial_d \left\{ {a \atop c\ b} \right\}' + \left\{ {a \atop c\ l} \right\}' \left\{ {l \atop b\ d} \right\}' - \left\{ {a \atop d\ l} \right\}' \left\{ {l \atop b\ c} \right\}',$$

$$K_{jdi}{}^h = \partial_d \left\{ {h \atop j\ i} \right\} - \partial_j \left\{ {h \atop d\ i} \right\} + \left\{ {h \atop l\ d} \right\} \left\{ {l \atop j\ i} \right\} - \left\{ {h \atop j\ l} \right\} \left\{ {l \atop d\ i} \right\},$$

and

$$S_{cjd}{}^h \stackrel{\text{def}}{=} B_c{}^j \partial_j \left\{ {h \atop d\ i} \right\} + B_c{}^j \left\{ {h \atop j\ l} \right\} \left\{ {l \atop d\ i} \right\} + B_d{}^j \partial_c \left\{ {h \atop j\ i} \right\} + B_d{}^j \left\{ {l \atop j\ i} \right\} \left\{ {h \atop l\ c} \right\}.$$

Waerden—Bortolotti derivative of (1. 10) yields

$$(2.3) \qquad '\nabla_d{}'\nabla_c B_b{}^h = H_{cbx}{}'\nabla_d C_x{}^h + C_x{}^h{}'\nabla_d H_{cbx}.$$

By changing c, d and subtracting it from the resulting equation so obtained, we get

$$(2.4) \quad '\nabla_d{}'\nabla_c B_b{}^h - '\nabla_c{}'\nabla_d B_b{}^h = C_x{}^h('\nabla_d H_{cbx} - '\nabla_c H_{dbx}) + H_{cbx}{}'\nabla_d C_x{}^h - H_{dbx}{}'\nabla_c C_x{}^h.$$

Using (1. 12), (2. 4) reduces to

$$(2.5) \qquad '\nabla_d{}'\nabla_c B_b{}^h - '\nabla_c{}'\nabla_d B_b{}^h =$$

$$= C_x{}^h('\nabla_d H_{cbx} - '\nabla_c H_{dbx}) - B_a{}^h(H_{cbx} H_{dx}{}^a - H_{dbx} H_{cx}{}^a) + C_y{}^h(H_{cbx} L_{dxy} - H_{dbx} L_{cxy}).$$

From (2. 2) and (2. 5) we get

$$(2.6) \qquad B_a{}^h{}'R^a{}_{dcb} + B_b{}^i B_c{}^j K_{jdi}{}^h - B_b{}^i S_{cjd}{}^h =$$

$$= C_x{}^h('\nabla_d H_{cbx} - '\nabla_c H_{dbx}) - B_a{}^h(H_{cbx} H_{dx}{}^a - H_{dbx} H_{cx}{}^a) + C_y{}^h(H_{cbx} L_{dxy} - H_{dbx} L_{cxy}).$$

Multiplying both sides of (2. 6) by $g_{kh} B_e{}^k$ and using (1. 7) and (1. 11)a, we obtain

$$(2.7) \quad 'g_{ea}('R^a{}_{dcb} + H_{cbx} H_{dx}{}^a - H_{dbx} H_{cx}{}^a) + B_b{}^i B_c{}^j B_e{}^k g_{kh} K_{jdi}{}^h - B_b{}^i B_e{}^k g_{kh} S_{cjd}{}^h = 0$$

or

$$(2.8) \qquad 'R_{dcbe} = H_{dbx} H_{cxe} - H_{cbx} H_{dxe} - B_b{}^i B_c{}^j K_{jdik} + B_b{}^i B_e{}^k S_{cjdk}.$$

(2. 8) will be called the "generalised Gauss characteristic equation" in a subspace of a Kähler space.

Again multiplying (2. 6) by $g_{kh} C_z{}^k$ and using (1. 11) a, b we get

$$(2.9) \qquad \delta_{xz}('\nabla_d H_{cbx} - '\nabla_c H_{dbx}) + \delta_{yz}(H_{cbx} L_{dxy} - H_{dbx} L_{cxy}) = 0$$

We call this the "generalised Mainardi Codazzi relation" in a subspace of a Kähler space.

3. *Some properties of the second fundamental tensor* H_{cbx}.

Let a holomorphic plane element determined by a certain vector u^h and its transform $F_i^h u^i$ by F_i^h at a point be given. We draw a totally geodesic subspace which is tangent to this holomorphic plane element and passes through this point.

We represent the subspace by

(3. 1) $$\xi^h = \xi^h(\eta^a)$$

where $a, b, c, d=1, 2$.

Then the subspace being totally geodesic, we have (1, p. 107)

(3. 2) $$\partial_c B_b^h + B_c^j B_b^i \begin{Bmatrix} h \\ j\ i \end{Bmatrix} - B_a^h \begin{Bmatrix} a \\ c\ b \end{Bmatrix}' = 0$$

substituting (3. 2) in (1. 9), we obtain

(3. 3) $$'\nabla_c B_b^h = 0$$

from which we have

Theorem 3. 1. *A necessary and sufficient condition for B_b^h in a subspace of a Kähler space to be totally geodesic is that its Waerden—Bortolotti derivative be zero*

Applying (3. 3) in (2. 5) we get

(3. 4)

$$0 = C_x^h('\nabla_d H_{cbx} - '\nabla_c H_{dbx}) - B_a^h(H_{cbx} H_{dx}{}^a - H_{dbx} H_{cx}{}^a) + C_y^h(H_{cbx} L_{dxy} - H_{dbx} L_{cxy}).$$

Multiplying both sides of (3. 4) by $g_{kh} B_e^k$ and using (1. 7) and (1. 11)a we obtain

$$'g_{ea}(H_{cbx} H_{dx}{}^a - H_{dbx} H_{cx}{}^a) = 0$$

or

(3. 5) $$H_{cbx} H_{dxe} - H_{dbx} H_{cxe} = 0$$

Thus we have

Theorem 3. 2. (3. 5) *is the necessary condition that a subspace of a Kähler space be totally geodesic.*

From (1. 11) and (3. 3) we get

(3. 6) $$H_{cbx} = 0$$

since we assume that $C_x^h \neq 0$.

Thus we have

Theorem 3. 3. *A necessary and sufficient condition for a subspace of a Kähler space to be totally geodesic is given by*

$$H_{cbx} = 0$$

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Reference

[1] K. YANO, Differential geometry on complex and almost complex spaces, *New York*, 1965.

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