Note on differential equations with constant coefficients

By RAHMI IBRAHIM IBRAHIM ABDEL KARIM (Cairo)

In a previous paper [1], we studied the resonance case in the differential equation of the n^{th} order

(a) $L[x] \equiv x^{(n)} + a_1(t)x^{(n-1)} + a_2(t)x^{(n-2)} + \cdots + a_{n-j}(t)x^{(j)} = f(t) \quad (1 \leqq j \leqq n-1),$

where the coefficients $a_\mu(t)$ $(\mu = 1, 2, \ldots, n-j)$ and $f(t)$ are continuous and periodic functions of the same period p and

$$a_{n-j}(t) \not\equiv 0. \tag{1}$$

In this note we shall assume that *all the coefficients $a_\mu(t)$ in* (a) *are constants and $f(t)$ has the period p.* The main results in [1] will be reduced in this case to interesting results.

Setting $\hat{x}(t) = x^{(j)}(t)$ in (a), we obtain the reduced differential equation of order $(n-j)$

$$(\hat{a}) \qquad \hat{L}[\hat{x}] \equiv \hat{x}^{(n-j)} + a_1\hat{x}^{(n-j-1)} + \cdots + a_{n-j}\hat{x} = f(t).$$

The homogeneous and adjoint equations corresponding to $(\hat{a})$ are

$$(\hat{b}) \qquad \hat{L}[\hat{y}] \equiv \hat{y}^{(n-j)} + a_1\hat{y}^{(n-j-1)} + \cdots + a_{n-j}\hat{y} = 0$$

and

$$(\hat{c}) \qquad \bar{L}[\hat{z}] \equiv (-1)^{n-j}\hat{z}^{(n-j)} + (-1)^{n-j-1}a_1\hat{z}^{(n-j-1)} + \cdots + a_{n-j}\hat{z} = 0$$

respectively.

We state the following lemma, which can be proved easily.

Lemma 1. *Under the assumption* (1), *all the p-periodic solutions of* $(\hat{b})$ *and* $(\hat{c})$ *have the mean value zero.*

We solve now the equation $(\hat{b})$ by means of the substitution $\hat{y}(t) = e^{\alpha^* t}$. Let α_ν^* $(\nu = 1, \ldots, s)$ be the pairwise distinct roots of the characteristic equation corresponding to $(\hat{b})$ with the multiplicities $\hat{m}_\nu$. Equation $(\hat{b})$ has p-periodic solutions iff between the characteristic roots α_ν^* (for $\nu = 1, \ldots, s$) there exists integral multivalues of $\frac{2\pi i}{p}$. Let the characteristic roots be arranged such that $\alpha_1^*, \ldots, \alpha_\varrho^*$ be integral multivalues of $\frac{2\pi i}{p}$, while the others are not. We notice that $\alpha_\nu^* \neq 0$, otherwise the equation $(\hat{b})$ possesses a constant solution, which contradicts with the assumption (1) (Lemma 1). We set $\alpha_\nu^* = \alpha_\nu + i\beta_\nu$ $(\nu=1, \ldots, \varrho, \ldots, s)$. The general solution of $(\hat{b})$ is

$$\hat{y}(t) = \sum_{\nu=1}^{s} \sum_{\mu=0}^{\hat{m}_\nu - 1} e^{\alpha_\nu^* t}\, {}^\nu c_\mu t^\mu,$$

where ${}^\nu c_\mu$ are arbitrary constants. Accordingly the first row of the fundamental matrix solution $\hat{Y}(t)$ of $(\hat{b})$, which represents the n linear independent solutions of $(\hat{b})$, has the form

$$\underline{\hat{y}}_1^T(t) = \left(\underline{\hat{y}}_{1,1}^T(t), \underline{\hat{y}}_{1,2}^T(t), \ldots, \underline{\hat{y}}_{1,s}(t)\right). \tag{2}$$

Here $(\nu=1, \ldots, s)$

$$\underline{\hat{y}}_{1,\nu}^T(t) = e^{i\beta_\nu t}\left(1, t, \ldots, \frac{t^{\hat{m}_\nu - 1}}{(\hat{m}_\nu - 1)!}\right) e^{\alpha_\nu t} = \tag{3}$$

$$= (e^{i\beta_\nu t}, O_2, \ldots O_{\hat{m}_\nu}) \cdot \begin{pmatrix} 1, & t, & \frac{t^2}{2!}, & \ldots, & \frac{t^{\hat{m}_\nu - 1}}{(\hat{m}_\nu - 1)!} \\ & 1, & t, & \ldots, & \frac{t^{\hat{m}_\nu - 2}}{(\hat{m}_\nu - 2)!} \\ & & \ddots & & \vdots \\ & & & & 1 \end{pmatrix} \cdot e^{\alpha_\nu t} = \underline{\hat{\varphi}}_{1,\nu}^T(t) e^{\hat{K}_\nu t},$$

where—as it can be easily verified (see e.g. [2] or [3]) —

$$\hat{K}_\nu = \begin{pmatrix} \alpha_\nu & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha_\nu \end{pmatrix} \quad \text{and} \quad \underline{\hat{\varphi}}_{1,\nu}^T(t) = (e^{i\beta_\nu t}, O_2, \ldots, O_{\hat{m}_\nu}). \tag{4}$$

Consequently the fundamental matrix solution $\hat{Y}(t)$ of $(\hat{b})$ is obtained in the required form

$$\hat{Y}(t) = \hat{\Phi}(t) e^{\hat{K}t} \tag{5}$$

(see [1], § 2), where the matrix $\hat{\Phi}(t)$ is p-periodic and the constant matrix $\hat{K}$ is in the Jordan canonical normal form with the submatrices $\hat{K}_\nu$ $(\nu=1, \ldots, s)$ of order $\hat{m}_\nu$. Denoting the elements of the first row of $\hat{\Phi}(t)$ by $\hat{\varphi}_\mu(t)$ (for $\mu = (\nu), (\nu)+1, \ldots, (\nu)+\hat{m}_\nu - 1$; $\nu=1, \ldots, \varrho, \ldots, s$), where the index (ν) is defined by

$$(\nu) = \sum_{\mu=1}^{\nu-1} m_\mu + 1, \tag{6}$$

we see from (4) that

$$\hat{\varphi}_{(\nu)}(t) = e^{i\beta_\nu t}, \quad \hat{\varphi}_\mu(t) \equiv 0 \text{ for } \mu = (\nu)+1, \ldots, (\nu)+\hat{m}_\nu - 1; \qquad \nu=1, \ldots, \varrho. \tag{7}$$

The fact that $\hat{y}_{(\nu)}(t) = \hat{\varphi}_{(\nu)}(t)$ $(\nu=1, \ldots, \varrho)$ possesses the mean value zero is also already proved in Lemma 1.

Thus we have proved the following

Theorem 1. *In the case of differential equations with constant coefficients, all the p-periodic functions* $\hat{\varphi}_\mu(t)$ *for* $\mu = (\nu), \ldots, (\nu)+\hat{m}_\nu - 1;\ \nu = 1, \ldots, \varrho$ *(see* (5) *and* (7)) *have the mean value zero, where* $\hat{\varphi}_\mu(t)$ *denotes the elements of the first row of the matrix* $\hat{\Phi}(t)$.

We turn now to the question of the power order of the solution $x(t)$ of (a) and its first $(n-1)$ derivatives.

It is well known [1], that in the resonance case[1]), under the essential condition (1), the solution $\hat{x}(t) = x^{(j)}(t)$ of $(\hat{a})$ and its derivatives $x^{(j+1)}, \ldots, x^{(n-1)}$ take on — independent of the initial values — values of the same minimal order

$$\hat{m} = \operatorname*{Max}_{\binom{\nu \text{ Resonance}}{\nu = 1, \ldots, \varrho}} (\hat{m}_\nu). \tag{8}$$

Further it is also shown in [1], that the minimal order of the solution $x(t)$ of (a) is

$$m = \operatorname*{Max}_{\binom{\nu \text{ Resonance}}{\nu = 0, 1, \ldots, \varrho}} (m_\nu), \tag{9}$$

where[2]) the resonance indices $\nu = 1, \ldots, \varrho$ are in both equations (a) and $(\hat{a})$ the same and $m_\nu = m_\nu(j)$ is obtained from the formula

$$\begin{cases} m_\nu = \hat{m}_\nu + \operatorname{Min}(j, i_{\nu+1}) - \operatorname{Min}(j, i_\nu) \quad \text{for} \quad (\nu = 1, \ldots, \lambda - 1) \\ m_\lambda = \hat{m}_\lambda + j - \operatorname{Min}(j, i_\lambda) \\ m_\nu = \hat{m}_\nu \quad \text{for} \quad (\nu = \lambda + 1, \ldots, s),\ m_0 = \operatorname{Min}(j, i_1). \end{cases} \tag{10}$$

The same statement holds also for the derivatives $x^{(k)}(t)$ $(k = 1, \ldots, j-1)$, if the index j is replaced by $j-k$ in the formula (10).

Referring to theorem 1 and the definition of the index i_ν and λ (see [1], § 3), we obtain the following corollaries:

1. In the case of differential equations with constant coefficients, there do not exist such indices i_ν.

2. $m_\nu = \hat{m}_\nu$ (for $\nu = 1, \ldots, \varrho, \ldots, s$), $m_0 = j$. This folows from (10) and corollary 1.

Referring to (9), it is required for $j > \hat{m}$ to know whether the index $\nu = 0$ is a resonance index or not.

We state the following

Lemma 2. $\nu = 0$ is a resonance index of the differential equation (a) iff the p-periodic function $f(t)$ has a mean value different from zero.

We evaluate the additional p-periodic solution $z_0(t)$ of the adjoint equation corresponding to (a) ([1], § 6). This p-periodic solution satisfies the inhomogeneous adjoint reduced differential equation

$$\hat{L}[\hat{u}] \equiv (-1)^{n-j} \hat{u}^{(n-j)} + (-1)^{n-j-1} a_1 \hat{u}^{(n-j-1)} + \cdots + a_{n-j} \hat{u} = 1 \tag{11}$$

[1]) For the definition of the resonance case or resonance index see [1], § 7.

[2]) For more investigations on the index $\nu = 0$, see [1], § 2 theorem 1 and § 6 theorem 8.

(see [4]), which possesses a particular solution $z_0^*(t) = \frac{1}{a_{n-j}}$. Thus $z_0(t)$ is uniquely determined up to an additive linear combination of the p-periodic solutions $\hat{z}(t)$ of ($\hat{c}$), which have — by virtue of Lemma 1 — the mean value zero. However it can be shown that z_0 is a constant.

Referring to the formula (9), corollary 2 and lemma 2, we obtain the following

Theorem 2. *If* $\int_0^p f(t)\,dt = 0$, *then the solution* $x(t)$ *of* (a) *and all its first* $(n-1)$ *derivatives take on in the resonance case — independent of the initial values — the same minimal power order, i.e.* t^m *with* $m = \hat{m} = \max\limits_{\binom{\nu \text{ Resonance}}{\nu=1,\dots,\varrho}} (\hat{m}_\nu)$. *Further if* $\int_0^p f(t)\,dt \neq$ $\neq 0$ *and simultaneously* $j > m$, *then the minimal power order of the derivatives* $x^{(k)}(t)$ $(k = 0, 1, \dots, j-m)$ *decreases monotonically by* 1 *starting from* $x(t)$ *with the minimal power order* t^j *till* $x^{(j-m)}(t)$ *with the minimal power order* t^m, *and remain from this value constant and equal* t^m. *Finally if* $\int_0^p f(t)\,dt \neq 0$ *and* $j \leqq m$, *then the first statement holds.*

References

[1] R. I. I. A. Karim, Studium des Resonanzfalles bei gewöhnlichen linearen reduzierten Differentialgleichungen mit periodischen Koeffizieten. *Acta U. Palackianae Olomuciensis F. R. N.* **27**, (1968), 27—49.

[2] R. Zurmühl, Matrizen und ihre technischen Anwendungen, 4. Aufl. *Berlin—Göttingen—Heidelberg* 1969.

[3] R. Bellmann, Introduction to matrix Analysis, *New York, Toronto, London* 1960.

[4] R. I. I. A. Karim, On the power order of the solutions and their derivatives of ordinary linear differential equations in the resonance case. *Acta F. R. N. U. Comen. Mathematica* **19**, (1968), 205—214.

Cairo University, Faculty of Science, Mathematical Departement, Cairo

(Received November 23, 1970.)