The minimum number of spanning paths in a strong tournament

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1. Introduction. A *tournament* T_n consists of a finite set of *nodes* $1, 2, \ldots, n$ such that each pair of distinct nodes i and j is joined by exactly one of the arcs $\vec{ij}$ or $\vec{ji}$. If the arc $\vec{ij}$ is in T_n we say that i *beats* j or j *loses* to i and write $i \to j$. If each node a subtournament A beats each node of a subtournament B we write $A \to B$ and let $A+B$ denote the tournament determined by the nodes of A and B.

A *path* is a sequence $P=\{p_1, p_2, \ldots, p_k\}$ of distinct nodes of T_n such that $p_i \to p_{i+1}$ if $1 \leqq i < k$; a *spanning* path contains every node of T_n. Let $h(T_n)$ denote the number of spanning paths in the tournament T_n. RÉDEI [1] showed that $h(T_n)$ is odd for every T_n and SZELE [2] showed that if $H(n)$ denotes the maximum of $h(T_n)$ over all tournaments with n nodes then

$$n!/2^{n-1} \leqq H(n) \leqq (n+1)!/2^{(3/4)n-3} \quad \text{for} \quad n \geqq 1.$$

A tournament T_n is *strong* if it cannot be expressed as $T_n = A+B$ for some nonempty subtournaments A and B. If a tournament T_n is not strong, or *weak*, it has a unique expression of the type $T_n = A+B+\cdots+K$ where the nonempty subtournaments $A, B, \ldots, K$ all are strong and $h(T_n) = h(A)\cdot h(B)\ldots h(K)$. Thus in considering lower bounds for $h(T_n)$ we may as well assume T_n is strong. Our object here is to prove the following result.

Theorem. *If $h(n)$ denotes the minimum number of spanning paths a strong tournament T_n can have, then $h(1)=1$ and*

$$\alpha^{n-1} \leqq h(n) \leqq \begin{cases} \beta^{n-1} & n \equiv 1 \pmod 3 \\ 9\cdot 5^{-4/3}\beta^{n-1} \text{ if } & n \equiv 2 \pmod 3 \\ 3\cdot 5^{-2/3}\beta^{n-1} & n \equiv 3 \pmod 3 \end{cases} \tag{1}$$

for $n \geqq 3$, where $\alpha = 6^{1/4} = 1.565$ and $\beta = 5^{1/3} = 1.710$.

2. An upper bound for $h(n)$. If $n \geqq 3$ let R_n denote the strong tournament in which $i \to j$ if and only if $i<j$ except that $n \to 1$. Any spanning path P of R_n that involves the arc $\vec{n1}$ partitions the nodes $(2, 3, \ldots, n-1)$ into two subsets, those that come before n and those that come after 1; it follows from the definition of R_n that the nodes in these two subsets must occur in natural order in the path P. Conversely, each partition of the nodes $(2, 3, \ldots, n-1)$ into two subsets determines a unique spanning path that involves the arc $\vec{n1}$. There is only one spanning path of R_n that doesn't

involve the arc $\overrightarrow{n1}$, namely, $\{1, 2, \ldots, n\}$. Consequently, $h(R_n) = 1+2^{n-2}$ for $n \geqq 3$.

More generally, let $\{s_1, s_2, \ldots, s_t\}$ denote an increasing subsequence of the nodes $1, 2, \ldots, n$ such that $s_1=1$, $s_t=n$, and $\alpha_i = s_{i+1}-s_i-1 \geqq 1$ for $1 \leqq i < t$. Let S_n denote the strong tournament in which $i \to j$ if and only if $i<j$ except that $s_{i+1} \to s_i$ for $1 \leqq i < t$. It is not difficult to show, by an extension of the argument just used for the case $t=2$, that

$$h(S_n) = \prod_{i=1}^{t-1} (1+2^{\alpha_i})$$

for $n \geqq 3$. (Notice that $h(S_n)$ is always odd, in accordance with Rédei's theorem.)

To minimize $h(S_n)$ for given values of n, notice that

$$(1+2^{\alpha+3}) > (1+2^2)(1+2^\alpha) \quad \text{for} \quad \alpha \geqq 1,$$

$$(1+2^3)^2 > (1+2^1)(1+2^2)^2,$$

and

$$(1+2^1)(1+2^3) = (1+2^1)^3 > (1+2^2)^2.$$

Thus if $n = 3k+1$ the minimum occurs when $t = k+1$ and all the α's equal two; if $n = 3k+2$ it occurs when $t = k+1$ and one α equals three and the rest equal two or when $t = k+2$ and two α's equal one and the rest equal two; if $n = 3k+3$ it occurs when $t = k+2$ and one α equals one and the rest equal two. It follows, therefore, that the minimum value of $h(S_n)$ is 5^k, $9 \cdot 5^{k-1}$, or $3 \cdot 5^k$ according as $n = 3k+1$, $3k+2$, or $3k+3$. This implies the upper bound in inequality (1).

3. A lower bound for $h(n)$. Before proceeding we remark that if node y is not contained in a path P and y beats some node x of P then y can be inserted in the path P before x; in particular, y can be inserted immediately before the first node of P it beats. Similarly, if y loses to x then y can be inserted in P after x.

It is not difficult to verify that $h(1)=1$, $h(3)=3$, $h(4)=5$, and $h(5)=9$ so the lower bound in inequality (1) certainly holds for small values of n. Suppose the inequality has been established for $3 \leqq n \leqq m-1$ and consider a strong tournament T_m such that $h(m)=h(T_m)$.

Let K denote any minimal subset of nodes of T_m whose removal, along with all incident arcs, leaves a weak subtournament R; such a subset K exists since, for example the removal of the nodes that lose to any given node of T_m leaves a weak subtournament. Since R is weak there exist subtournaments A, B and C where A and C are strong such that $R = A+B+C$; it may be that B is vacuous. Let a, b, c and k denote the number of nodes in A, B, C and K so that $a+b+c+k = m$. It follows from the minimality property of K and the fact that T_m is strong that each node of K loses to at least one node of C and beats at least one node of A. We now construct two families of spanning paths of T_m and thus obtain a lower bound for $h(m)$.

Let P_1, P_2, and P_3 denote any spanning paths of A, B, and C; there are at least $h(a)h(c)$ spanning paths of R and they all are of the type $P_1+P_2+P_3$. Now let X and Y denote any partition of the nodes of K into two subsets. Since each node of X beats some node of A and each node of Y loses to some node of C it follows that the nodes of X and Y can be inserted in the path $P_1+P_2+P_3$ so as to form a spanning path Q of T_m that first passes through the nodes of X and A, then passes through the nodes of B, and finally passes through the nodes of Y and C. Different paths P_1 and P_3 and different partitions X and Y yield different paths Q so T_m must have at least $2^k h(a)h(c)$ spanning paths of this first type.

Let x denote any given node of K and let u and v denote given nodes of A and C that lose to x and beat x, respectively. For any spanning path P_1 of A let P_{11} denote the (possibly empty) subpath determined by the nodes that come before u in P_1 and let P_{12} denote the subpath determined by u and the nodes that come after u in P_1; for any spanning path P_3 or C let P_{31} denote the subpath determined by v and the nodes that come before v in P_3 and let P_{32} denote the (possibly empty) subpath determined by the nodes that come after v in P_3; for any partition of the nodes of B into two subsets W and Z, let P_{21} and P_{22} denote spanning paths of the subtournaments determined by W and Z. It is not difficult to see that for each partition W and Z there are at least $h(a)h(c)$ paths of the type $P^* = P_{11}+P_{21}+$ $+P_{31}+\{x\}+P_{12}+P_{22}+P_{32}$ that pass through x and the nodes of R. We now estimate the number of ways of inserting the remaining nodes of K into such a path P^*.

If $k \geqq 2$, let y denote any node of $K-\{x\}$. Since y beats some node of A it follows that y can be inserted in P^* before the last node or P_{12}; it can also be inserted arter the last node of P_{12} unless it beats that node and all the nodes of P_{22} and P_{32}. Since y loses to some node of C it follows that y can be inserted in P^* arter the first node of P_{31}; it can also be inserted before the first node of P_{31} unless it loses to that node and to all nodes of P_{11} and P_{21}. Thus if y can't be inserted in P^* in at least two ways it must be that $y \to Z$ and $W \to y$, among other things. Notice that this last statement also applies when P^* is replaced by any path obtained by inserting other nodes of $K-\{x\}$ in P^*.

Let the partitions of B into two subsets be numbered from $i=1$ to $i=t=2^b$. For the i-th partition, W_i and Z_i, let γ_i denote the number of nodes y of $K-\{x\}$ such that $y \to Z_i$ and $W_i \to y$. It follows from the observations in the last paragraph that there are at least $2^{k-1-\gamma_i}$ ways to insert the nodes of $K-\{x\}$ in any path P^* associated with the partition W_i and Z_i so as to form a spanning path of T_m. If we apply this construction for all spanning paths of A and C and for all partitions of B we obtain at least

$$(2) \qquad 2^{k-1}(2^{-\gamma_1}+\dots+2^{-\gamma_t})h(a)h(c)$$

different spanning paths of T_m. (Some node of C comes before some node of A in each of these paths so they are not the same as any of the paths enumerated earlier.) Since the nodes of B that beat and lose to any node y of $K-\{x\}$ are fixed, each such node y contributes one to just one γ_i and $\gamma_i+\dots+\gamma_t = k-1$. Therefore, the sum in (2) is minimized when all the γ_i's equal $(k-1)2^{-b}$ and T_m must have at least $h(a)h(c)2^{b+(k-1)(1-2^{-b})}$ spanning paths of this second type.

It follows from these two constructions that $h(m) \geqq (2^k+2^{b+(k-1)(1-2^{-b})})h(a)h(c)$. In order to show that this implies that $h(m) \geqq \alpha^{m-1} = \alpha^{a+b+c+k-1}$, it suffices to show, in view of the induction hypothesis, that

$$(3) \qquad 2^k+2^{b+(k-1)(1-2^{-b})} \geqq \alpha^{b+k+1} = 6^{(b+k+1)/4}$$

for all integers b and k such that $b \geqq 0$ and $k \geqq 1$.

Since $\log_2 \alpha < \frac{2}{3}$, inequality (3) certainly holds if $k \geqq 2(b+k+1)/3$ or $k \geqq 2b+2$ and it can easily be verified directly for $1 \leqq k \leqq 2b+1$ when $0 \leqq b \leqq 2$. (Notice that equality holds in (3) when $k=1$ and $b=2$.) If $b \geqq 3$, then inequality (3) holds if

$$2^k + 2^{b+7(k-1)/8} \geqq \alpha^{b+k+1} \tag{4}$$

for $b \geqq 3$ and $k \geqq 1$. This inequality certainly holds if $b+7(k-1)/8 \geqq 2(b+k+1)/3$ or $5k+8b \geqq 37$; if $b \geqq 4$ this holds for $k \geqq 1$ and if $b=3$ it holds if $5k \geqq 13$ or if $k \geqq 3$. In the remaining cases, when $b=3$ and $k=1$ or 2, inequality (4) can be verified directly. This suffices to complete the proof of the lower bound of inequality (1) by induction.

References

[1] L. RÉDEI, Ein kombinatorischer Satz, *Acta Litt. Szeged*, **7** (1934), 39—43.

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