Syntopogenous structures and complete regularity

By D. V. THAMPURAN (Stony Brook, N.Y.)

An extensive theory of syntopogenous structures has been developed by CSÁSZÁR [1]. The purpose of this paper is to study the relationship between syntopogenous structures and complete regularity.

A syntopogenous structure gives rise to two topologies in general and so it is natural to associate a bitopological space with a syntopogenous structure. A syntopogenous structure characterizes a particular type of bitopological space; a bitopological space is completely regular iff it is syntopogenizable. This is similar to the result that a symmetric syntopogenous structure characterizes a particular type of topological space — the completely regular one.

The bitopological space of a perfect syntopogenous structure has some special properties.

Let E denote a set. The empty subset of E will be denoted by $\emptyset$ and for a subset A of E we will write cA for the complement of A. If A contains only one element x we will write cx for cA.

Definition 1. Let $\mathscr{I}$, $\mathscr{I}'$ be two topologies for E. Then the ordered triple $(E, \mathscr{I}, \mathscr{I}')$ is said to be a bitopological space; $\mathscr{I}$ and $\mathscr{I}'$ are called the left and right topologies of this space.

Let $<$ be a topogenous structure on E as defined on page 59 of CSÁSZÁR (1). Denote by $\mathscr{I}$ the family of all subsets T of E such that $x \in T$ implies $cT < cx$; it is then obvious that $\mathscr{I}$ is a topology for E. Take $\mathscr{I}'$ to be the family of all subsets T of E such that $x \in T$ implies $x < T$; then $\mathscr{I}'$ is also a topology for E.

Definition 2. We will call $(E, <, \mathscr{I}, \mathscr{I}')$ the bitopological space, $\mathscr{I}$ the left topology and $\mathscr{I}'$ the right topology of $<$.

But when the context makes the meaning clear we will also denote this space by $(M, <)$ or M.

Denote by k, k' the Kuratowski closure functions respectively for $\mathscr{I}, \mathscr{I}'$. Express composition of functions by juxtaposition; thus ck will denote $c(kA)$ for all subsets A of E. Take $i = ckc$, $i' = ck'c$; then i and i' are the interior functions for k and k'. We will also write (E, k, k') for the bitopological space $(E, \mathscr{I}, \mathscr{I}')$. When A contains only a single point x we will write kx for kA and $k'x$ for $k'A$.

Theorem 1. *Let A be a subset of E. Then*

$$iA = \{x : cA < cx\} \quad \textit{and} \quad i'A = \{x : x < A\}.$$

PROOF. Let $B=\{x:cA<cx\}$. Then $iA\subset B\subset A$ and so $iA=B$ if $iB=B$. Let $x\in B$. Then $cA<cx$ and so there is $C\subset E$ such that $cA<C<cx$. Now $y\in cC$ implies $C\subset cy$ and so $cA<cy$; hence $y\in B$ which implies $cC\subset B$. Therefore $cB\subset C$ whence $cB<cx$ and so $iB=B$. The other part of the theorem can be proved similarly.

Corollary. $kA=\{xA\not< cx\}$ and $k'A=\{x:x\not< cA\}$.

Theorem 2. *$A<B$ iff $kA<i'B$.*

PROOF. Let $A<B$. Then $kA\subset B$ since $x\in kA$ implies $x\in B$ for $x\in cB$ implies $B\subset cx$ and so $A<cx$ which is a contradiction. Also $x\in A$ implies $x<B$ and so $x\in i'B$ from which it follows $A\subset i'B$.

Now $A<B$ implies there is $C\subset E$ such that $A<C<B$. Hence $kA\subset C<B$ and so $kA<B$. Also $A<C\subset i'B$ and so $A<i'B$. Thus $A<B$ implies $kA<B$ which in turn implies $kA<i'B$.

The converse is obvious.

Each topogeneous structure $<$ gives rise to a topogenous structure $<'$ defined by $A<'B$ iff $cB<cA$. It is easy to see that the left and right topologies of $<'$ coincide respectively with the right and left topologies of $<$. Hence $<'$ generates no new topologies.

Definition 3. Let $(E, \mathcal{I}, \mathcal{I}')$, $(N, \mathcal{N}, \mathcal{N}')$ be two bitopological spaces and f a function from E to N. Then f is said to be continuous iff is $\mathcal{I}-\mathcal{N}$ continuous and $\mathcal{I}'-\mathcal{N}'$ continuous.

Definition 4. Let $(E, \mathcal{I}, \mathcal{I}')$ be a bitopological space and F a subset of E. Denote by $\mathcal{N}$, $\mathcal{N}'$ the relativizations respectively of $\mathcal{I}$, $\mathcal{I}'$ to N. Then $(N, \mathcal{N}, \mathcal{N}')$ is said to be a subspace of $(E, \mathcal{I}, \mathcal{I}')$.

Let R be the set of all real numbers. Define a quasimetric m for R as follows: for all real x, y,

$$m'(x, y) = \begin{cases} y-x, & x\leqq' y \\ 0, & y<' x \end{cases}$$

where $<'$ denotes the usual order for the reals. For subsets A, B of R write $M(A, B)=$ $=\inf\{m'(x, y):x\in A, y\in B\}$. Define $<^*$ for R by $A<^*B$ iff $M(A, cB)>0$. Then $<^*$ is a topogenous structure on R.

Definition 5. We will call $<^*$ the usual topogenous structure on R and the bitopological space of $<^*$ the usual bitopological space for R. If A is a subset of R then the subspace for A is said to be the usual bitopological space for A. Denote by (R, r, r') the usual bitopological space for R. Let I denote the closed unit interval $[0, 1]$ of the reals. We will also denote by I the usual bitopological space for I.

Definition 6. A bitopological space $(E, \mathcal{I}, \mathcal{I}')$ is said to be completely regular iff

(i) A is $\mathcal{I}$-closed and y in cA imply there is a continuous function f from E to I such that $fA=0$ and $f(y)=1$ and
(ii) B is $\mathcal{I}'$-closed and $x\in cB$ imply there is a continuous function g from E to I such that $g(x)=0$ and $gB=1$.

For x, y in R let $xR=\{y:x<'y\}$ and $Rx=\{y:y<'x\}$ where $<'$ is the usual order relation for the reals. Then the set of all xR for x in R is a base for the left topology of R and the set of all Rx for x in R is a base for the right topology of R.

$<'$ will denote the usual order relation for the reals in Lemmas 1, 2 and Theorem 3. Lemma 1. is well known.

Lemma 1. *For each t in a dense subset D of the positive reals let $S(t)$ be a subset of E such that*

(i) $S(t)\subset S(u)$ if $t<'u$ *and*
(ii) $\cup\{S(t):t\in D\}=E$.

For x in E take $f(x)=\inf\{t:x\in S(t)\}$. Then

$$\{x:f(x)<'u\}=\cup\{S(t):t\in D \quad\text{and}\quad t<'u\}$$

and

$$\{x:f(x)\leqq'u\}=\cap\{S(t):t\in D \quad\text{and}\quad u<'t\}$$

for every real u.

Lemma 2. *Let (E, k, k') be a bitopological space. For each t in a dense subset D of the positive reals let $S(t)$ be a subset of E such that*

(i) $i'S(t)=S(t)$
(ii) $kS(t)\subset S(u)$ if $t<'u$ *and*
(iii) $\cup\{S(t):t\in D\}=E$.

Then the function f from E to R defined by $f(x)=\inf\{t:x\in S(t)\}$ is continuous.

Proof. For a real u the set $f^{-1}Ru=\{x:f(x)<'u\}$ is the union of i'-open sets and so is i'-open. Hence f is $k'-r'$ continuous.

Next, for a real u, the set $f^{-1}uR=\{x:u<'f(x)\}$ and so $cf^{-1}uR=\{x:f(x)\leqq'u\}=$ $=\cap\{S(t):t\in D, u<'t\}=A$, say. Now $A\subset\cap\{kS(t):t\in D, u<'t\}$. Also

$$\cap\{kS(t):t\in D, u<'t\}\subset A$$

since $t\in D$, $u<'t$ imply there is v in D such that $u<'v<'t$ and so $kS(v)\subset S(t)$. Hence A is the intersection of k-closed sets and so is k-closed. Therefore $icA=cA$ and this implies f is $k-r$ continuous.

Theorem 3. *Let $(E, <, k, k')$ be a bitopological space and let $A<B$. There is then a continuous function f from E to I such that $fA=0$ and $fcB=1$.*

Proof. Let D be the set of all numbers of the form $p2^{-q}$ where p and q are positive integers. Take $S(t)=E$ for t in D and $1<'t$, take $S(1)=B$ and take $S(0)$ to be an i'-open set such that $A<S(0)<B$. For t in D and $0<'t<'1$ take t in the form $t=(2m+1)2^{-n}$ and choose, inductively on n, $S(t)$ to be an i'-open set such that $S(2m2^{-n})<S(t)<S((2m+2)2^{-n})$. Such choice is possible since $<$ is a topogenous structure. Take $f(x)=\inf\{t:x\in S(t)\}$. Then f is continuous. Also $fA=0$ and $fcB=1$.

Corollary. $A<B$ implies there is a continuous function f from E to I such that $fkA=0$ and $fk'cB=1$.

Corollary. The bitopological space $(E, <, k, k)$ is completely regular.

Let S be a syntopogenous structure for E. Define $\mathscr{I}$ to be the family of all subsets T of E such that x in T implies $cT<'cx$ for some $<'$ in S. Then $\mathscr{I}$ is a topology for E. Similarly the family $\mathscr{I}'$, of all subsets T of E such that x in T implies $x<'T$ for some $<'$ in S, is also a topology for E.

Definition 7. We will say $(E, S, \mathscr{I}, \mathscr{I}')$ is the bitopological space of S, $\mathscr{I}$ is the left topology of S and $\mathscr{I}'$ is the right topology of S.

Given a syntopogenous structure S on E define a binary relation $<$ by $A<B$ iff $A<'B$ for some $<'$ in S. Then $<$ is a topogenous structure on E and the left and right topologies of $<$ coincide respectively with the left and right topologies of S. Hence a syntopogenous space (E, S) is completely regular. Also $A<'B$ for some $<'$ in S implies there is a continuous function f from E to I such that $fA=0$ and $fcB=1$.

Definition 8. A bitopological space $(E, \mathscr{I}, \mathscr{I}')$ is said to be syntopogenizable (or topogenizable) iff there is a syntopogenous structure (or topogenous structure) on E whose bitopological space is $(E, \mathscr{I}, \mathscr{I}')$.

THAMPURAN [3] has proved that a completely regular bitopological space is quasiuniformizable. From a quasiuniformity $\mathscr{U}$ we can get a syntopogenous structure S — in the same way as a symmetric syntopogenous structure can be obtained from a uniformity — such that $\mathscr{U}$ and S have the same bitopological space.

It is clear that a bitopological space is topogenizable iff it is syntopogenizable. We now have the result:

Theorem 4. *A bitopological space is completely regular iff it is topogenizable.*

It is obvious that a subspace of a completely regular space is completely regular. Thampuran (2) has proved that a product of completely regular spaces is completely regular.

Definition 9. A bitopological space (M, k, k') is said to be regular iff

(i) $A=kA$ and $y\in cA$ imply there are sets $X=iX$, $X'=i'X'$ such that $A\subset X'$ and $y\in X$ and $X\cap X' = \emptyset$ and
(ii) $B=k'B$ and $x\in cB$ imply there are sets $Y=iY$ $Y'=i'Y'$ such that $x\in Y'$ and $B\subset Y$ and $Y\cap Y' = \emptyset$.

A completely regular space is evidently regular; hence a syntopogenous space is regular. It is clear that a subspace of a regular space is regular. A product of regular spaces has been shown to be regular by Thampuran (2).

Theorem 5. *Let $<$ be a perfect topogenous strucutre on E. Then*

(i) $A \nless B$ *iff* $A\cap k'cB \neq \emptyset$ *and*
(ii) $kA = \cup \{kx : x\in A\}$.

PROOF.

(i) $A\nless B$ implies there is x in A such that $x\nless B$ and so x is in $k'cB$. Conversely, if there is x in A such that x is in kcB then $x\nless B$ and hence $A\nless B$.
(ii) $x\in kA$ iff $A\nless cx$ and this holds iff there is y in A such that $y\nless cx$ or $x\in ky$.

We also have the following result for a perfect topogenous structure $<$ on E. If $<'$ is also a topogenous structure on E such that both $<$ and $<'$ have the same bitopological space (E, k, k') then $<$ is finer than $<'$, $A < cx$ implies $A <' cx$ and $x < cA$ implies $x <' cA$. But if $<'$ is also perfect then $< = <'$. These follow easily from Theorem 5., and from Corollary to Theorem 1.

References

[1] Á. CSÁSZÁR, Foundations of general topology, *New York*, 1963.
[2] D. V. THAMPURAN, Bitopological spaces and complete regularity *(to appear)*.
[3] D. V. THAMPURAN, Bitopological spaces and quasiuniformities *(to appear)*.

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