Publ. Math. Debrecen
49 / 1-2 (1996), 177–182

Points on the plane whose coordinates are terms of recursive sequences

By KÁLMÁN LIPTAI (Eger)

Abstract. Let $\{R_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ $(n = 0, 1, 2, \dots)$ be sequences of integers defined by $R_n = AR_{n-1} - BR_{n-2}$ and $V_n = AV_{n-1} - BV_{n-2}$, where A and B are fixed non-zero integers. We give a condition when the distance from the points $P_n(R_n, V_n)$ to the line $y = \sqrt{D}x$ tends to zero. Moreover we show that there is no lattice point (x, y) nearer than $P_n(R_n, V_n)$ if and only if $|B| = 1$.

Let $\{R_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ be second order linear recurring sequences of integers defined by

$$\begin{aligned} R_n &= AR_{n-1} - BR_{n-2} \quad (n > 1),\\ V_n &= AV_{n-1} - BV_{n-2} \quad (n > 1), \end{aligned}$$

where $A > 0$ and B are fixed non-zero integers and the initial terms of the sequences are $R_0 = 0,\ R_1 = 1,\ V_0 = 2$ and $V_1 = A$. Let α and β be the roots of the characteristic polynomial $x^2 - Ax + B$ of these sequences and denote by D its discriminant. Then we have

$$\sqrt{D} = \sqrt{A^2 - 4AB} = \alpha - \beta, \quad A = \alpha + \beta, \quad B = \alpha\beta. \tag{1}$$

Throughout the paper we suppose that $D > 0$ and D is not a perfect square. In this case, α and β are two irrational real numbers and $|\alpha| \neq |\beta|$,

---

Mathematics Subject Classification: 11 B 39.

Research supported by the Hungarian National Scientific Research Foundation, Operating Grant Number OTKA T 016 975 and the Foundation for Hungarian Higher Education and Research.

so we can suppose that $|\alpha| > |\beta|$. Furthermore, as it is well known, the terms of the sequences R and V are given by

$$R_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad \text{and} \quad V_n = \alpha^n + \beta^n. \tag{2}$$

From these equations it is not difficult to see that

$$\lim_{n\to\infty} \frac{R_{n+1}}{R_n} = \alpha \quad \text{and} \quad \lim_{n\to\infty} \frac{V_n}{R_n} = \alpha - \beta = \sqrt{D} \tag{3}$$

(see, e.g. [3], [7]).

J. P. Jones and P. Kiss [4] considered the points $P_n = (R_n, R_{n+1})$, from a geometric point on view, as lattice points on the Euclidean plane. Using related results on the diophantine approximation of α they investigated, how the points P_n approach the line $y = \alpha x$, as $n \to \infty$. They proved that the distance of the points P_n from this line tends to zero as $n \to \infty$ if and only if $|\beta| < 1$. They obtained similar results in the three-dimensional case, too. G. E. Bergum [1] and A. F. Horadam [2] showed that the points $P_n = (x, y)$ lie on the conic section $Bx^2 - Axy + y^2 + eB^n = 0$, where $e = AR_0R_1 - BR_0^2 - R_1^2$ and the initial terms R_0 and R_1 are not necessarily 0 and 1. For the Fibonacci sequence, when $A = 1$ and $B = -1$, C. Kimberling [6] characterized conics satisfied by infinitely many Fibonacci lattice points $(x, y) = (F_m, F_n)$.

In this paper we investigate the geometric properties of the lattice points $P_n = (R_n, V_n)$. We shall use the following result of J. P. Jones and P. Kiss [5]: If $|B| = 1$ and $B + 5 \le A$, then all rational solutions p/q of the inequality

$$\left| \sqrt{D} - \frac{p}{q} \right| < \frac{2}{\sqrt{D} q^2}$$

are of the form $p/q = V_n/R_n$ for some positive integer n, if q is sufficiently large.

Let us consider the points $P_n = (R_n, V_n)$ $(n = 1, 2, \dots)$ on the plane. Then (3) shows that the slope of the vector OP_n tends to $\sqrt{D}$. But it is not obvious that the points P_n approach the line $y = \sqrt{D}x$, as $n \to \infty$. The following theorem shows a condition for this.

Theorem 1. *Let d_n be the distance from the point $P_n = (R_n, V_n)$ to the line $y = \sqrt{D}x$. Then $\lim_{n\to\infty} d_n = 0$ if and only if $|\beta| < 1$.*

Proof. The distance d_{x_0,y_0} from a point (x_0, y_0) to the line $y = \sqrt{D}x$ is given by

$$d_{x_0,y_0} = \left| \frac{\sqrt{D}x_0 - y_0}{\sqrt{D+1}} \right|. \tag{4}$$

Thus, using (4), we have

$$(5)\qquad d_n = \left|\frac{\sqrt{D}R_n - V_n}{\sqrt{D+1}}\right| = \left|\frac{\sqrt{D}\frac{\alpha^n-\beta^n}{\alpha-\beta} - (\alpha^n+\beta^n)}{\sqrt{D+1}}\right| = \frac{2|\beta|^n}{\sqrt{D+1}},$$

from which the theorem follows.

We note that $|\beta| < 1$ holds when $|B+1| < |A|$.

The previous theorem implies that the points P_n converge to the line $y = \sqrt{D}x$ if $|\beta| < 1$, but these lattice points P_n are not necessarily the nearest lattice points to $y = \sqrt{D}x$ in all cases. Let $d_{x,y}$ denote the distance between the lattice point (x,y) and the line $y = \sqrt{D}x$, and let d_n be the distance defined in the previous theorem. We prove

Theorem 2. *If n is sufficiently large and $B+5 \le A$, then there is no lattice point (x,y) such that $d_{x,y} \le d_n$ and $|x| < |R_n|$ if and only if $|B| = 1$.*

Proof. First suppose $|B| = 1$. In this case, obviously, $|\beta| < 1$ and α is irrational as it was supposed. Assume that for some n there is a lattice point (x,y) such that $d_{x,y} \le d_n$ and $|x| < |R_n|$. Then, by (4) and (5)

$$\left|\sqrt{D}x - y\right| < 2|\beta|^n$$

follows.

From this, using (2) and the fact that $|\alpha\beta| = |B| = 1$, we obtain the inequalities

$$(6)\qquad \left|\sqrt{D} - \frac{y}{x}\right| \le \frac{2|\beta|^n}{|x|} = \frac{2}{|\alpha|^n|x|} = \frac{2\left|1-\left(\frac{\beta}{\alpha}\right)^n\right|}{\sqrt{D}|R_n x|} < \frac{2\left|1-\left(\frac{\beta}{\alpha}\right)^n\right|}{\sqrt{D}x^2}.$$

By the above mentioned result of J. P. Jones and P. Kiss and its proof we get that (6) holds only if $x = R_i$ and $y = V_i$ for some i. So $x = R_i$ is a term of the sequence R. The sequence R is a nondegenerate one with $D > 0$ and $|B| = 1$. So it can be easily seen that $|R_t|, |R_{t+1}|, \dots$ is an increasing sequence if t is sufficiently large. Furthermore by (5), $d_k > d_j$, if $k < j$.

Thus, $i < n$ and $d_i > d_n$ follows, which contradicts $d_i = d_{x,y} \le d_n$. So the first part of the theorem is proved.

To complete the proof, we have to prove that if $|B| > 1$, then there are lattice points (x,y) such that $d_{x,y} < d_n$ and $|x| < |R_n|$ for any sufficiently large n.

Suppose $|B| > 1$. If $|\beta| > 1$, then, by (5), $d_n \to \infty$ as $n \to \infty$, so there are lattice points (x, y) such that $d_{x,y} < d_n$ and $|x| < |R_n|$ for any sufficiently large n.

If $|\beta| = 1$, then d_n is a constant and there are infinitely many n and points (x, y) such that $d_{x,y} \le d_n$ and $|x| < |R_n|$.

If $|\beta| < 1$, then by (2) and $|B| > 1$, we have

$$\left|\sqrt{D} - \frac{V_n}{R_n}\right| = \frac{2|\beta|^n}{|R_n|} = \frac{2|B|^n \left|1 - \left(\frac{\beta}{\alpha}\right)^n\right|}{\sqrt{D}R_n^2} > \frac{Q}{R_n^2} \tag{7}$$

for any fixed $Q > 0$ if n is sufficiently large. In this case, $\sqrt{D}$ is an irrational number.

It is known that if y/x is a convergent of the continued fraction expansion of $\sqrt{D}$, then

$$\left|\sqrt{D} - \frac{y}{x}\right| < \frac{1}{2|x|^2}. \tag{8}$$

In (8) let y, and hence x, be large enough and let the index n be defined by $|R_{n-1}| \le |x| < |R_n|$.

From (4), (5), (7) and (8) we obtain the inequalities

$$d_n > \frac{Q}{|R_n|\sqrt{D+1}} \quad \text{and} \quad d_{x,y} < \frac{1}{2|x|\sqrt{D+1}}.$$

So we have $d_{x,y} < d_n$ with $|x| < |R_n|$ because

$$\frac{Q}{|R_n|} = \frac{Q}{|R_{n-1}\alpha|(1-(\beta/\alpha)^n)/(1-(\beta/\alpha)^{n-1})} > \frac{1}{2|R_{n-1}|} \ge \frac{1}{2|x|}.$$

This completes the proof.

Lastly, we give equations that are satisfied by the lattice points (R_n, V_n).

Theorem 3. *All lattice points $(x, y) = (R_n, V_n)$ satisfy one of the equations*

$$y = \sqrt{D}x + c(x)|x|^{\delta} \tag{i}$$

or

$$y = \sqrt{D}x - c(x)|x|^{\delta}, \tag{ii}$$

where $\delta = \log|\beta|/\log|\alpha|$ *and* $c(x)$ *is a function such that* $\lim_{x\to\infty} c(x) = 2\left(\sqrt{D}\right)^{\delta}$.

Proof. By (2), we have

$$V_n = \alpha^n + \beta^n = R_n\sqrt{D} + 2\beta^n \tag{9}$$

and

$$|R_n| = \frac{|\alpha|^n}{\sqrt{D}}(1 - (\beta/\alpha)^n). \tag{10}$$

From (10), we have

$$n = \frac{\log|R_n| + \log\sqrt{D} - \varepsilon n}{\log|\alpha|}$$

where $\varepsilon_n = \log(1 - (\beta/\alpha)^n)$ and $\varepsilon_n \to 0$ as $n \to \infty$ since $|\beta/\alpha| < 1$. This implies that

$$\begin{aligned} \beta^n &= \pm\exp\left\{\frac{\log|\beta|\log|R_n|}{\log|\alpha|} + \frac{\log|\beta|\log\sqrt{D}}{\log|\alpha|} - \frac{\varepsilon_n\log|\beta|}{\log|\alpha|}\right\} \\ &= \pm|R_n|^{\delta}\sqrt{D}^{\delta_n}, \end{aligned} \tag{11}$$

where $\delta = \log|\beta|/\log|\alpha|$ and

$$\delta_n = \frac{\log|\beta|}{\log|\alpha|} - \frac{\varepsilon_n\log|\beta|}{\log\sqrt{D}\log|\alpha|} \to \delta \quad \text{as} \quad n \to \infty, \tag{12}$$

since $\varepsilon_n \to 0$ as $n \to \infty$.

From (9), (11) and (12), the theorem follows.

Remark. The lattice points (R_n, V_n) satisfy (i) for every n if $\beta > 0$. If $\beta < 0$, then the lattice points satisfy (i) and (ii) alternately.

References

[1] G. E. Bergum, Addenda to Geometry of a generalized Simson's Formula, *Fibonacci Quart.* **22** No. 1 (1984), 22–28.

[2] A. F. Horadam, Geometry of a Generalized Simson's Formula, *Fibonacci Quart.* **20** No. 2 (1982), 164–68.

[3] D. Jarden, Recurring Sequences, *Riveon Lematematika, Jerusalem (Israel)*, 1958.

[4] J. P. Jones and P. Kiss, On points whose coordinates are terms of a linear recurrence, *Fibonacci Quart.* **31** No. 3 (1993), 239–245.

[5] J. P. Jones and P. Kiss, Some diophantine approximation results concerning linear recurrences, *Math. Slovaca* **42** No. 5 (1992), 583–591.

[6] C. Kimberling, Fibonacci Hyperbolas, *Fibonacci Quarterly* **28** No. 1 (1990), 22–27.
[7] E. Lucas, Theorie des fonctions numériques simplement periodiques, *American J. Math.* **1** (1978), 184–240, 289–321.

Kálmán Liptai
Eszterházy Károly Teachers' Training College
Department of Mathematics
H–3301 Eger, P.O.Box 43
Hungary

(Received January 23, 1996)