On the independence of the axiom of choice

By ALEXANDER ABIAN (Ames, Iowa)

One of the main reasons for difficulty in proving the independence [1] and the relative consistency [2] of the set-theoretical axioms is the axiom scheme of Replacement. The specific difficulty in connection with the proof of the independence of the axiom of Choice is mentioned in [3, p. 537]. On the other hand, however, even with the exclusion of the axiom scheme of Replacement, the proof of the mutual independence of the remaining set-theoretical axioms (including the axiom of Choice) is nontrivial and is of considerable interest. Accordingly, in this paper we prove the relative consistency of the set-theoretical axioms of *Extensionality*, *Powerest*, *Sumset*, *Infinity* and *Choice* (i.e., all the usual set-theoretical axioms except the axiom scheme of Replacement) and the independence of each of these axioms from the remaining four.

Without invoking axioms other than those mentioned here, it is believed that perhaps the independence of each of the axioms of Sumset and Choice from the remaining four, *cannot be proved in a better way than that given here* (see our proofs of Lemmas 4 and 6).

Our proofs employ transfinite recursion based on the ordinal numbers [4, p. 316] of the Zermelo—Fraenkel Theory of Sets (abbreviated by ZF). As usual, by ZF we mean a first order theory without equality whose one and only nonlogical symbol is the *elementhood symbol* "$\in$" and whose nonlogical axioms are the axioms of *Extensionality*, *Powerset*, *Sumset*, *Infinity* and the *axiom scheme of Replacement*. We also observe that, as shown in [5], the consistency of the axioms of Extensionality, Powerset, Sumset and Choice and the independence of each of these axioms from the remaining three can be proved without resorting to transfinite recursion and in a manner which requires almost no knowledge of the formal Theory of Sets.

In what follows the *equality sign* "$=$" is introduced by its usual set-theoretical definition (i.e., $x=y$ if and only if every element of x is an element of y and vice versa). The set-theoretical indistinguishability between equal sets is secured by the axiom of Extensionality, which states that *equal sets are elements of the same sets*. The axiom of Powerset states that every set has a powerset (i.e., a set whose elements are exactly the subsets of s). The axiom of Sumset states that *every set has a sumset* (i.e., a set whose elements are exactly the elements of the elements of s). As usual, $\emptyset$ denotes the *empty set* (i.e., $x\notin\emptyset$, for every x). Moreover, we let x^+ denote the *successor* of x (i.e., $y\in x^+$ if and only if $y\in x$ or $y=x$). Accordingly, the axiom of Infinity states that *there exists a set t such that $\emptyset\in t$ and if $x\in t$ then $x^+\in t$*. If a set w has a unique element in common with every nonempty element of a nonempty set s, and if w has no other elements, then w is called a *choice-set* of s. The choice-set of the empty set is defined to be the empty set. With this in mind, the axiom of Choice states that *every disjointed set has a choice-set*, where a set d is called disjointed if and only if $\emptyset \notin d$ and two distinct elements of d have no element in common.

When no confusion is likely to arise, we write

$$a=\{\ldots\} \tag{1}$$

where inside the braces we insert exactly all those sets which are elements of a.

As expected, we assume that ZF is consistent and by virtue of the Completeness theorem, we let $(K, \in)$ denote a model for ZF.

In what follows, by a "*set*" we mean an object of the model $(K, \in)$, and, unless otherwise specified, by "*is an element of*" we refer to "$\in$" appearing in $(K, \in)$. Naturally, all other set-theoretical items are introduced by their usual definitions. Also every mentioned ordinal number is an object of the model $(K, \in)$.

Lemma 1. *The axioms of Extensionality, Powerset, Sumset, Infinity and Choice form a consistent system of axioms.*

PROOF. Consider the ordinal number $\omega 2$ of the model $(K, \in)$. Using notation (1), we have:

$$\omega 2 = \{0, 1, 2, \ldots, \omega, \ \omega+1, \omega+2, \ldots, \omega+n, \ldots\} \tag{2}$$

where n denotes a natural number and where

$$\begin{gathered} 0=\emptyset, \quad 1=\{0\}, \quad 2=\{0, 1\}, \quad 3=\{0, 1, 2\}, \ldots \\ \omega=\{0, 1, 2, \ldots\}, \quad \omega+1 = \{0, 1, 2, \ldots, \omega\}, \ldots \\ \omega+n+1 = \{0, 1, 2, \ldots, \omega, \ \omega+1, \ldots, \omega+n\}, \ldots . \end{gathered} \tag{3}$$

Next, consider the model $(\omega 2, \in)$ whose domain of individuals is $\omega 2$, given by (2), and whose elementhood relation $\in$ is that of the model $(K, \in)$.

We show that $(\omega 2, \in)$ is a model for the axioms of Extensionality, Powerset, Sumset, Infinity and Choice.

As shown by (2) and (3), since no two differently denoted sets are equal, we see that the axiom of Extensionality is valid in $(\omega 2, \in)$.

Let us observe that the only subsets of, say, 1 that can be found in $(\omega 2, \in)$ are 0 and 1. Therefore, 2, as shown by (3), is the set whose elements are exactly the subsets of 1 that can be found in $(\omega 2, \in)$. Consequently, 2 is the powerset of 1 in $(\omega 2, \in)$. Similarly, it is easy to verify that in $(\omega 2, \in)$.

$$u+1 \quad \textit{is the powerset of} \quad u$$

for every set u (where, as expected, u denotes an ordinal number less than $\omega 2$). Thus, the axiom of Powerset is valid in $(\omega 2, \in)$.

Next, we observe that, say, 2 in $(\omega 2, \in)$ is the set whose elements are exactly the elements of the elements of 3. Therefore, 2 is the sumset of 3 in $(\omega 2, \in)$. Similarly, it is easy to verify that in $(\omega 2, \in)$,

$$0 \textit{ is the sumset of } 0$$

$$\omega \textit{ is the sumset of } \omega$$

$$u-1 \textit{ is the sumset of } u$$

for every set u such that $u \neq 0$ and $u \neq \omega$. Thus, the axiom of Sumset is valid in $(\omega 2, \in)$.

Clearly, in $(\omega 2, \in)$

$$u+1 \textit{ is the successor of } u$$

for every set u. Hence ω in $(\omega 2, \in)$ is such that $0 \in \omega$ and if $x \in \omega$ then $x^+ \in \omega$. Thus, the axiom of Infinity is valid in $(\omega 2, \in)$.

Finally, we observe that, except for the empty set there is no disjointed set in $(\omega 2, \in)$. Thus, the axiom of choice is valid in $(\omega 2, \in)$.

Hence, Lemma 1 is proved.

Lemma 2. *The axiom of Extensionality is independent of the axioms of Powerset, Sumset, Infinity and Choice.*

Proof. Consider the model $(\omega 2, \in^*)$ whose domain of individuals is $\omega 2$, given by (2), and whose elementhood relation $\in^*$ is defined by

(4)
$x \in^* y$ *if and only if* $x \in y$
for every set y, *where* y *denotes an*
ordinal number less than $\omega+1$.
Otherwise,
$x \in^* y$ *if and only if* $x \in (y-1)$.

Since the *equality* relation is defined in terms of the *elementhood* relation, in view of (4), we let $=^*$ denote the equality symbol in the model $(\omega 2, \in^*)$.

Clearly, by virtue of (2), (3) and (4), we have in $(\omega 2, \in^*)$,

$$\omega =^* (\omega+1)$$

whereas,

$$\omega \in^* (\omega+2) \quad \textit{and} \quad (\omega+1) \notin^* (\omega+2)$$

But the above implies that the axiom of Extensionality is not valid in $(\omega 2, \in^*)$.

Again, in view of (2), (3) and (4), it is easy to verify that in $(\omega 2, \in^*)$,

$$n+1 \textit{ is the powerset of } n$$

for every set n, where n denotes a natural number, and

$$u+3 \textit{ is the powerset of } u$$

for every set u, where u denotes an infinite ordinal number.

Thus, the axiom of Powerset is valid in $(\omega 2, \in^*)$.

Also, in view of (2), (3) and (4) it is easy to verify that in $(\omega 2, \in^*)$,

$$0 \quad \textit{is the sumset of} \quad 0$$

$$\omega \quad \textit{is the sumset of} \quad \omega \quad \text{and} \quad \omega+1$$

$$u-1 \quad \textit{is the sumset of} \quad u$$

for every set u, where u denotes an ordinal number distinct from 0, ω and $\omega+1$.

Thus, the axiom of Sumset is valid in $(\omega 2, \in^*)$.

From (2), (3) and (4) it follows that in $(\omega 2, \in^*)$,

$$n+1 \quad \textit{is the successor of} \quad n$$

for every set n, where n denotes a natural number. Also, (2) shows that ω is a set of $(\omega 2, \in^*)$.

Thus, the axiom of Infinity is valid in $(\omega 2, \in^*)$.

Since, except for the empty set, there is no disjointed set in $(\omega 2, \in^*)$ we see that the axiom of Choice is valid in $(\omega 2, \in^*)$.

Hence, Lemma 2 is proved.

Lemma 3. *The axiom of Powerset is independent of the axioms of Extensionality, Sumset, Infinity and Choice.*

PROOF. Consider the model $(\omega+1, \in)$ whose domain of individuals is $\omega+1$ given by (3) and whose elementhood relation $\in$ is that of the model $(K, \in)$.

Since there is no set in the model $(\omega+1, \in)$ whose element is ω, we see that the powerset of ω does not exist in $(\omega+1, \in)$. Thus, the axiom of Powerset is not valid in $(\omega+1, \in)$. On the other hand, following the lines of reasoning in the proof of Lemma 1, we see that the axioms of Extensionality, Sumset, Infinity and Choice are valid in $(\omega+1, \in)$.

Hence, Lemma 3 is proved.

Next, we prove the independence of the axiom of Sumset from the remaining four axioms by constructing a model recursively.

Lemma 4. *The axiom of Sumset is independent of the axioms of Extensionality, Powerset, Infinity and Choice.*

PROOF. Let u be an ordinal number and A_u a set (naturally, of the model $(K, \in)$). For every element m of A_u, we let $P_u(m)$ denote the set of precisely those subsets of m that can be found in A_u, i.e.,

$$(5) \qquad x \in P_u(m) \quad \textit{if and only if} \quad x \in A_u \quad \textit{and} \quad x \subset m$$

We refer to $P_u(m)$ as "*the powerset of m relativized to* A_u".

Thus, if

$$(6) \qquad A_0 = \{\{1, 2\}, \{1, 3\}, \{\{1, 2\}, \{1, 3\}\}, 0, 1, 2, \ldots, \omega\}$$

then the powerset $P_0(\{1, 2\})$ of $\{1, 2\}$ relativized to A_0 is $\{0, \{1, 2\}\}$ since the only subsets of $\{1, 2\}$ that can be found in A_0 are 0 and $\{1, 2\}$.

Next, based on (5), we let:

(7) $$P_u = \{x | x = P_u(m) \quad \textit{for some} \quad m \in A_u\}$$

We observe that the existence of P_u as a set of the model $(K, \in)$ is ensured by the existence of A_u and the fact that $(K, \in)$ is a model for ZF.

Now, for every ordinal u, based on (6) and (7) and by virtue of Transfinite Induction, we define the set A_u by:

(8) $$\begin{aligned} A_0 &= \{\{1, 2\}, \{1, 3\}, \{\{1, 2\}, \{1, 3\}\}, 0, 1, 2, \ldots, \omega\} \\ A_{u+1} &= A_u \cup P_u \\ A_u &= \bigcup_{v<u} A_v \quad \textit{if } u \textit{ is a limit ordinal number.} \end{aligned}$$

Let us consider the model $(A, \in)$ defined as follows:

(9) $$\begin{gathered} x \quad \textit{is a set of} \quad (A, \in) \quad \textit{if and only if} \\ x \in A_u \quad \textit{for some ordinal number} \quad u \end{gathered}$$

where A_u is given by (8). We note that, as indicated in (9), the elementhood relation "$\in$" in $(A, \in)$ is the same as in $(K, \in)$. We show that in the model $(A, \in)$ the axioms of Extensionality, Powerset, Infinity and Choice are valid, whereas the axiom of Sumset is not valid.

From (8) it follows that $y \in x$ in $(A, \in)$ if and only if $y \in x$ in $(K, \in)$. Consequently, $x = y$ in $(A, \in)$ if and only if $x = y$ in $(K, \in)$. However, since the axiom of Extensionality is valid in $(K, \in)$, we see that the axiom of Extensionality is also valid in the model $(A, \in)$.

For every set x of $(K, \in)$, we let

(10) $$r(x)$$

denote the *smallest ordinal number* such that

(11) $$x \in A_{r(x)}. \quad \textit{Otherwise}, \quad r(x) = 0.$$

Let s be a set of $(A, \in)$ and let $\mathfrak{P}(s)$ be the powerset of s in $(K, \in)$. Using notation (1), we let

(12) $$p = \text{lub}\ \{r(x) | x \in \mathfrak{P}(s)\}$$

From (5), (7), (8), (11) and (12) it follows that there exists an element $P(s)$ of A_{p+1} such that $P(s)$ is the set of all the subsets of s that can be found in $(A, \in)$. Thus, the axiom of Powerset is valid in the model $(A, \in)$.

From (8) it follows that $\{1, 2, 3\}$ is not an element of A_0. Moreover, since 0 is an element of every relativized powerset, again, from (8) it follows that $\{1, 2, 3\}$ is not an element of A_u for every ordinal number u. Consequently, $\{1, 2, 3\}$ is not a set of $(A, \in)$. Hence the set $\{\{1, 2\}, \{1, 3\}\}$ of $(A, \in)$ has no sumset in $(A, \in)$. Thus, the axiom of Sumset is not valid in the model $(A, \in)$.

Since ω and all its elements are elements of A_0, we see that the axiom of Infinity is valid in the model $(A, \in)$.

Also, since there is no disjointed set in $(A, \in)$, other than $\emptyset$, the axiom of Choice is valid in the model $(A, \in)$.

Hence, Lemma 4 is proved.

Lemma 5. *The axiom of Infinity is independent of the axioms of Extensionality, Powerset, Sumset and Choice.*

Proof. Consider the model $(\omega, \in)$ whose domain of individuals is ω given by (3) and whose elementhood relation $\in$ is that of the model $(K, \in)$. Following the lines of reasoning in the proof of Lemma 1, we see that except for the axiom of Infinity all the other axioms mentioned in the Lemma are valid in the model $(\omega, \in)$.

Lemma 6. *The axiom of Choice is independent of the axioms of Extensionality, Powerset, Sumset and Infinity.*

Proof. Let u be an ordinal number and B_u a set (naturally of the model $(K, \in)$). For every element m of B_u, we let $S_u(m)$ denote the set of precisely those elements of the elements of m that can be found in B_u, i.e.,

$$x \in S_u(m) \quad \textit{if and only if} \quad x \in B_u \quad \textit{and} \quad y \in B_u \tag{13}$$
$$\textit{and} \quad x \in y \quad \textit{and} \quad y \in m.$$

As expected, we refer to $S_u(m)$ as "*the sumset of m relativized to* B_u".

Next, based on (13), we let

$$S_u = \{x \mid x = S_u(m) \ \textit{for some} \ m \in B_u\} \tag{14}$$

We observe that the existence of S_u as a set of the model $(K, \in)$ is ensured by the existence of B_u and the fact that $(K, \in)$ is a model for ZF.

Now, for every ordinal u, by virtue of Transfinite Induction, we define the set B_u by:

$$B_0 = \{\{2,\} \ \{\{2, \{2\}\}, \ 0, 1, 2, \dots, \omega\}$$
$$B_{u+1} = B_u \cup P_u \cup S_u \tag{15}$$
$$B_u = \bigcup_{v<u} B_v \quad \textit{if} \quad u \quad \textit{is a limit ordinal number}$$

where P_u is as given by (5) and (7) with A_u replaced by B_u and where S_u is given by (14).

Let us consider the model $(B, \in)$ defined as follows:

$$x \quad \textit{is a set of} \quad (B, \in) \quad \textit{if and only if} \tag{16}$$
$$x \in B_u \quad \textit{for some ordinal number} \quad u$$

where B_u is given by (15). We note that, as indicated in (16), the elementhood relation "$\in$" in $(B, \in)$ is the same as in $(K, \in)$. We show that in the model $(B, \in)$ the axioms of Extensionality, Powerset, Sumset and Infinitely are valid, whereas the axiom of Choice is not valid.

The proof of the fact that the axioms of Extensionality, Powerset and Infinity are valid in the model $(B, \in)$ can be given analogously to that for the case of the model $(A, \in)$, given in the proof of Lemma 4, with A everywhere replaced by B.

Next, we observe that in view of (13), (14) and (15), if x is a set of $(B, \in)$ then the set of exactly all the elements of the elements of x (i.e., the sumset of x) is an element of $B_{r(x)+1}$ where $r(x)$ is given by (10) and (11). Thus, the axiom of Sumset is valid in the model $(B, \in)$.

Next, by (5) and (15), for every ordinal u and every $m \in B_u$ we have $m \in P_u(m)$. Hence, by (13), for every ordinal w, we have

$$w > u \quad \textit{implies} \quad S_w(P_u(m)) = m$$

But then, from the above and (15), it follows that

$$B_{u+1} = B_u \cup P_u \cup S_u = B_u \cup P_u \tag{17}$$

Finally, we prove that the axiom of choice is not valid in the model $(B, \in)$ by showing that the set $\{2, \{2\}\}$ of the model has no choice-set in the model. Indeed, the only possible choice-sets of $\{2, \{2\}\}$ are $\{1, 2\}$ and $\{0, 1\}$. However, $\{1, 2\} \notin B_0$. Moreover, as (5), (7), (15) and (17) show, for $u > 0$, every element x of B_u is a relativized powerset and therefore $0 \in x$. Consequently, $\{1, 2\}$ is not a set of the model $(B, \in)$. On the other hand, $\{0, 2\} \notin B_0$ and therefore, $\{0, 2\}$ must be a relativized powerset of some set m of the model $(B, \in)$. But this implies that $m=0$ or $m=2$. However, $m=0$ is impossible since $0 \neq \{0, 2\}$. Similarly, $m=2$ is impossible since $1 \notin \{0, 2\}$. Consequently, $\{0, 2\}$ is also not a set of the model $\{B, \in\}$, Thus, $\{2, \{2\}\}$ has no choice-set in the model $(B, \in)$ and the Lemma is proved.

We observe that all pur proofs in the above are given within ZF. Therefore, based on Lemmas 1 to 6 we have:

Theorem *The relative consistency of the axioms of Extensionality, Powerset, Sumset, Infinity and Choice as well as the independence of each from the remaining ones can be proved within ZF.*

References

[1] P. J. Cohen, The Independence of the Continuum Hypothesis, I, II. *Proc. Nat. Acad. Sci. U.S.A.* **50**, (1963), 1143—1148; **51**, (1965), 105—110.

[2] K. Goedel, The Consistency of the Axiom of Choice and the Generalized Continuum Hypothesis with the Axioms of Set Theory. *Princeton*, 1958.

[3] P. J. Cohen, A Minimal Model for Set Theory, *Bull. Amer. Math. Soc.* **69**, (1963), 537—540.

[4] A. Abian, The Theory of Sets and Transfinite Arithmetic. *Philadelphia*, Pa, 1965.

[5] A. Abian, On the Independence of Set-Theoretical Axioms. *Amer. Math. Month.*, **76**, (1969), 787—790.

(Received January 11, 1971.)