Symmetric generalized topological structures II.

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IV

Uniform convergence and convergence in proximity

Let X be a set. Let (Y, δ) be a symmetric generalized proximity space, and let $(Y, \mathscr{U})$ be a symmetric generalized uniform space. Let $\{f_n | n \in D\}$ be a net of members of Y^X, and let $f \in Y^X$.

(4. 1) *Definition.* (S. LEADER [18].) *f_n converges to f in proximity with respect to δ* iff for every $A \in P(X)$ and $B \in P(Y) f[A] \bar{\delta} B$ implies there exists m_0 such that if $n \geqq m_0$ then $f_n[A] \bar{\delta} B$.

(4. 2) *Definition. f_n converges to f uniformly with respect to $\mathscr{U}$* iff for every $U \in \mathscr{U}$ there exists m_0 such that for every $x \in X f_n(x) \in U[f(x)]$ if $n \geqq m_0$.

(4. 3) **Theorem.** *If f_n converges in proximity to f with respect to δ, then f_n converges to f uniformly with respect to $\mathscr{U}_2(\delta)$.*

PROOF. Let $V = (U_{A_1, B_1} \cap \dots \cap U_{A_n, B_n}) \in \mathscr{U}_2(\delta)$. It is sufficient to show that $f_n(x)$ is eventually in $V[f(x)]$ for all $x \in X$. Clearly, if $x \in X$ and

$$f(x) \in Y - \cup \{(A_i \cup B_i) | i = 1, \dots, n\},$$

then $V[f(x)] = Y$. Let $\{E_1, \dots, E_m\}$ be the family of all residual intersections of the A_i and B_i which have a non-void intersection with the range of f. For each c $c = 1, \dots, m$ let $F_c = f^{-1}(E_c)$. Let $x \in F_c$ where $1 \leqq c \leqq m$. Then $f(x) \in E_c$. We may assume that $E_c \subset E_c^* = (A_{k_1} \cap \dots \cap A_{k_p} \cap B_{j_1} \cap \dots \cap B_{j_q})$ for some $k_1, \dots, k_p; j_1, \dots$ $\dots, j_q$, and E_c intersects no other A_i or B_i. Consequently by Lemma (3. 6) and DeMorgan's law we have that $V(f(x)) = (Y - H_c)$ where $H_c = B_{k_1} \cup \dots \cup B_{k_p} \cup$ $\cup A_{j_1} \cup \dots \cup A_{j_q})$; so that by Lemma (3. 17) $E_c^* \bar{\delta} H_c$. But then $E_c \bar{\delta} H_c$ and since $f(F_c) \subset E_c$, we have that $f[F_c] \bar{\delta} H_c$. Thus by hypothesis there exists n_{E_c} such that for all $n \geqq n_{E_c} f_n[F_c] \bar{\delta} H_c$; so that $f_n[F_c] \subset (Y - H_c)$. Consequently, for every $n \geqq n_{E_c}$ we have that $f_n(x) \in V(f(x))$ for every $x \in F_c$. Choose n^* such that $n^* \geqq n_{E_c}$ for $c = 1, \dots$ $\dots, m$. Then for every $n \geqq n^*$ we have that $f_n(x) \in V[f(x)]$ for every $x \in X$.

(4. 4) *Remark.* In [18] S. Leader has considered nets of functions on a set X to a proximity space (Y, δ), and has given a definition of uniform convergence of such a net in terms of a certain family $\mathscr{P}$ of pseudometrics. It can be shown that the uniformity $\mathscr{U}$ generated by the totally bounded pseudometrics in this family $\mathscr{P}$ is in the

proximity class $\pi^*(\delta)$ of symmetric uniformities on X, and hence $\mathcal{U}=\mathcal{U}_2(\delta)$. Thus Theorem 15 in [18] provides a converse of our Theorem (4. 3) when δ is a classical proximity. More precisely this result states that if (Y, δ) is a proximity space and $\{f_n|n\in D\}$ is a net of members of Y^X and $f\in Y^X$, then f_n converges to f uniformly respect with to $\mathcal{U}_2(\delta)$ implies that f_n converges in proximity to f with respect to δ.

Note that (4. 3) holds good also in the case if $f_n[A]\cap B=\varnothing$ is replaced by $f_n[A]\bar{\delta}B$ in the definition (4. 1) and this modified definition follows from the uniform convergence of f_n to f regarding certain $\mathcal{U}$ compatible with the δ.

V

Uniform continuity and *P*-continuity

(5. 1) *Definition.* Let $(X, \mathcal{U}_a)$ and $(Y, \mathcal{U}_b)$ be symmetric generalized uniform spaces. A map $f:(X, \mathcal{U}_a)\to(Y, \mathcal{U}_b)$ is *uniformly continuous* iff for every $U\in\mathcal{U}_b$ there is a $V\in\mathcal{U}_a$ such that $(f(x), f(y))\in U$ provided $(x, y)\in V$. f is a *uniform isomorphism* iff it is 1—1, onto, and it, as well as its inverse, is uniformly continuous.

(5. 2) *Definition.* Let (X, δ_a) and (Y, δ_b) be symmetric generalized proximity spaces. A map $f:(X, \delta_a)\to(Y, \delta_b)$ is *p-continuous* iff $A\delta_a B$ implies $f(A)\delta_b f(B)$ for all A, B in $P(X)$. f is a *p-isomorphism* iff it is 1—1, onto, and it, as well as its inverse, is p-continuous.

(5. 3) **Theorem.** *Let (X, δ_a) and (Y, δ_b) be symmetric generalized proximity spaces. If $f:(X, \delta_a)\to(Y, \delta_b)$ is p-continuous, then $f:(X, \mathcal{I}(\delta_a))\to(Y, \mathcal{I}(\delta_b))$ is continuous.*

Proof. Let $A\subset X$. It is sufficient to show that $f(\bar{A})\subset\overline{f(A)}$. Let $x\in\bar{A}$. Then $x\delta_a A$; so that $f(x)\delta_b f(A)$. Hence $f(x)\in\overline{f(A)}$.

(5. 4) **Theorem.** *Let $(X, \mathcal{U}_a)$ and $(Y, \mathcal{U}_b)$ be symmetric generalized uniform spaces. If $f:(X, \mathcal{U}_a)\to(Y, \mathcal{U}_b)$ is uniformly continuous, then $f:(X, \delta(\mathcal{U}_a))\to(Y, \delta(\mathcal{U}_b))$ is p-continuous.*

Proof. Let $A\subset X$, $B\subset X$. Suppose $A\delta(\mathcal{U}_a)B$. Let U be any member of $\mathcal{U}_b$. Since f is uniformly continuous, there exists $V\in\mathcal{U}_a$ such that if $(x, y)\in V$, then $(f(x), f(y))\in U$. But since $A\delta(\mathcal{U}_a)B$ we have by Theorem (2. 27a) that there exists $t_1\in A$, $t_2\in B$ such that $(t_1, t_2)\in V$. Consequently, $(f(t_1), f(t_2))\in U$; so that again by Theorem (2. 27a) we have that $f(A)\delta_b f(B)$.

(5. 5) **Theorem.** *Let $(X, \mathcal{U}_a)$ and $(Y, \mathcal{U}_b)$ be symmetric generalized uniform spaces. If $f:(X, \mathcal{U}_a)\to(Y, \mathcal{U}_b)$ is uniformly continuous, then $f:(X, \mathcal{I}(\mathcal{U}_a))\to(Y, \mathcal{I}(\mathcal{U}_b))$ is continuous.*

Proof. We know by Theorem (2. 2) that there exist symmetric generalized proximities δ_a on X and δ_b on Y such that $\mathcal{U}_a\in\pi(\delta_a)$ and $\mathcal{U}_b\in\pi(\delta_b)$. By Theorem (5. 3) and Theorem (5. 4) we have that $f:(X, \mathcal{I}(\delta_a))\to(Y, \mathcal{I}(\delta_b))$ is continuous. But by Remark (2. 10) we have that $\mathcal{I}(\mathcal{U}_a)=\mathcal{I}(\delta_a)$ and $\mathcal{I}(\mathcal{U}_b)=\mathcal{I}(\delta_b)$.

(5. 6) **Theorem.** *Let* $(X, \mathscr{U}_a)$ *and* $(Y, \mathscr{U}_b)$ *be uniform spaces and let* $\mathscr{U}_b$ *be totally bounded. If* $f:(X, \delta(\mathscr{U}_a)) \to (Y, \delta(\mathscr{U}_b))$ *is p-continuous, then* $f:(X, \mathscr{U}_a) \to (Y, \mathscr{U}_b)$ *is uniformly continuous.*

(5. 7) *Remark.* W. J. THRON on page 202 in [40] has given a proof of Theorem (5. 6) which he ascribes to J. L. HURSCH. The next theorem is a partial generalization of Theorem (5. 6).

(5. 8) **Theorem.** *Let* $(X, \mathscr{U})$ *be a symmetric generalized uniform space. Let* (Y, δ) *be a symmetric generalized proximity space with proximity class* $\pi(\delta)$. *Let* $\mathscr{U}_1(\delta)$ *(as constructed in Theorem* (2. 23)*) be the least element in* $\pi(\delta)$. *If* $f:(X, \delta(\mathscr{U})) \to (Y, \delta)$ *is p-continuous, then* $f:(X, \mathscr{U}) \to (Y, \mathscr{U}_1(\delta))$ *is uniformly continuous.*

PROOF. Suppose $U \in \mathscr{U}_1(\delta)$. Since $\{U_{C,D} | C\bar{\delta}D\}$ is a base for $\mathscr{U}_1(\delta)$ there exist C, D in $P(Y)$ such that $C\bar{\delta}D$ and $U \supset U_{C,D}$. Since f is p-continuous, $f^{-1}(C)\overline{\delta(\mathscr{U})}f^{-1}(D)$. Let $V = U_{f^{-1}(C), f^{-1}(D)}$. Clearly, if $(x, y) \in V$, then $(f(x), f(y)) \in U_{C,D}$. But $V \in \mathscr{U}_1(\delta(\mathscr{U}))$ and since $\mathscr{U} \supset \mathscr{U}_1(\delta(\mathscr{U}))$, it follows that f is uniformly continuous.

(5. 9) *Remark.* Let $(X, \mathscr{U}_a)$ be a compact symmetric uniform space. Let $(Y, \mathscr{U}_b)$ be a correct uniform space. Then $f:(X, \mathscr{I}(\mathscr{U}_a)) \to (Y, \mathscr{I}(\mathscr{U}_b))$ is continuous implies $f:(X, \mathscr{U}_a) \to (Y, \mathscr{U}_b)$ is uniformly continuous.

A proof of this fact can be given which is verbatim the same as the proof for the well-known classical theorem in which $(X, \mathscr{U}_a)$ and $(Y, \mathscr{U}_b)$ are uniform spaces (cf. [40] page 187). The reason for this is that a uniform space has a symmetric base and that the proof does not use the fact that $\mathscr{U}_b$ satisfies (M. 5).

VI

Completeness

In this chapter the concept of completeness is defined for a symmetric generalized uniform space. A number of theorems are proved to indicate that the definition is a proper one. Also, it is shown that every separated correct uniform space has a completion.

We first list all of the relevant definitions. Unless otherwise noted, $(X, \mathscr{U})$ will denote a symmetric generalized uniform space.

(6. 1) *Definition.* A non-void family $\mathscr{J}$ of subsets of a set X is called a *filter* on X iff it satisfies the three conditions:

(F. 1) $A \in \mathscr{J}$, $B \in \mathscr{J}$ imply $(A \cap B) \in \mathscr{J}$.

(F. 2) $B \supset A \in \mathscr{J}$ implies $B \in \mathscr{J}$.

(F. 3) $\emptyset \notin \mathscr{J}$.

(6. 2) *Definition.* x_0 is a *limit* of the filter $\mathscr{J}$ iff $\mathscr{J} \supset \mathscr{N}(x_0)$, the neighborhood system of x_0. If x_0 is a limit of the filter $\mathscr{J}$, then we say that $\mathscr{J}$ *converges* to x_0 and that $\mathscr{J}$ is a convergent filter.

(6. 3) *Definition.* x_0 is a *cluster point* of the filter $\mathcal{F}$ iff every neighborhood of x_0 intersects every element of the filter.

(6. 4) *Definition.* A non-void family $\mathcal{B}$ of subsets of a set X, is a *filter base* on X provided $\mathcal{B}$ does not contain the null set and provided the intersection of any two elements of $\mathcal{B}$ contains an element of $\mathcal{B}$.

(6. 5) *Definition.* Let $(X, \mathcal{U})$ be a symmetric generalized uniform space. A filter $\mathcal{F}$ on X is *weakly Cauchy* with respect to $\mathcal{U}$ iff for every $U \in \mathcal{U}$ there exists $x \in X$ such that $U[x] \in \mathcal{F}$. $\mathcal{F}$ is *Cauchy* with respect to $\mathcal{U}$ iff for every $U \in \mathcal{U}$ there exists $A \in \mathcal{F}$ such that $(A \times A) \subset U$.

(6. 6) *Remark.* For a discussion of the history of the weakly Cauchy filter concept see [39].

(6. 7) *Definition.* A Cauchy filter in $(X, \mathcal{U})$ is an *infrafilter* iff it does not properly contain a Cauchy filter.

(6. 8) *Definition.* $(X, \mathcal{U})$ is *complete* iff every weakly Cauchy filter on X has a cluster point in X.

(6. 9) *Definition.* $(X, \mathcal{U})$ is *Δ-complete* iff whenever $(X, \mathcal{U})$ is uniformly isomorphic to a dense subspace $(X_a, \mathcal{U}_a)$ of $(X_b, \mathcal{U}_b)$, then $X_a = X_b$.

(6. 10) *Remark.* In a similar way we define a Δ-complete (separated) correct uniform space by taking $\mathcal{U}_a$ and $\mathcal{U}_b$ to be (separated) correct uniformities.

(6. 11) *Definition.* A symmetric generalized uniform space $(X_b, \mathcal{U}_b)$ is a *completion* of the symmetric generalized uniform space $(X, \mathcal{U})$ iff $(X_b, \mathcal{U}_b)$ is complete, and $(X, \mathcal{U})$ is uniformly isomorphic to a dense subspace $(X_a, \mathcal{U}_a)$ of $(X_b, \mathcal{U}_b)$.

(6. 12) *Remark.* The reader should recall the following facts about convergence theory. (A) If $\mathcal{B}$ is a filter base, then the family $\mathcal{F}(\mathcal{B})$ consisting for all sets A such that $A \supset B$ for some $B \in \mathcal{B}$ is a filter. (B) Every filter is contained in an ultrafilter. (C) If $\mathcal{F}$ is an ultrafilter and $\bigcup \{A_i | i=1, \dots, n\} \in \mathcal{F}$, then at least one $A_i \in \mathcal{F}$. (D) An ultrafilter converges to each of its cluster points. (E) $(X, \mathcal{T})$ is compact iff every filter on X has a cluster point in X. (F) $(X, \mathcal{T})$ is compact iff every ultrafilter on X converges to some point in X. (G) A Cauchy filter converges to each of its cluster points if $(X, \mathcal{U})$ is a correct uniform space.

(6. 13) *Remark.* In the usual statement of (6. 12G), $(X, \mathcal{U})$ is taken to be a uniform space. However, the proof is the same if we assume that $(X, \mathcal{U})$ is a correct uniform space since the proof does not require (M. 5).

(6. 14) **Theorem.** *Every Cauchy filter on $(X, \mathcal{U})$ is weakly Cauchy.*

PROOF. Let $U \in \mathcal{U}$. There exists $F \in \mathcal{F}$ such that $(F \times F) \subset U$. Let $x_0 \in F$. Then $F \subset U[x_0]$; so that $U[x_0] \in \mathcal{F}$.

(6. 15) **Theorem.** *If $(X, \mathcal{T})$ is a symmetric, connected topological space, then there exists a totally bounded symmetric generalized uniformity $\mathcal{U}$ on X such that $\mathcal{T}(\mathcal{U}) = \mathcal{T}$, and every filter in X is weakly Cauchy with respect to $\mathcal{U}$.*

PROOF. We know by Corollary (1. 14) that there exists a symmetric generalized proximity δ on X such that $\mathcal{I}(\delta)=\mathcal{I}$. Let $\mathcal{U}_1(\delta)$ be the uniformity on X that we constructed in Theorem (2. 23). $\mathcal{U}_1(\delta)\in\pi(\delta)$; so that $\mathcal{I}(\mathcal{U}_1(\delta))=\mathcal{I}$. Let $U\in\mathcal{U}_1(\delta)$. Then there exists sets $A\subset X$ and $B\subset X$ such that $U\supset U_{A,B}\supset U_{\bar{A},B}$. But since $\mathcal{I}$ is connected, there exists $x_0\in(X-(\bar{A}\cup\bar{B}))$; so that $U_{\bar{A},B}[x_0]=X$. Hence every filter on X is weakly Cauchy with respect to $\mathcal{U}$.

(6. 16) *Remark*. Theorem (6. 15) points out that it is not reasonable to require every weakly Cauchy filter in a symmetric generalized uniform space to converge in order for the space to be "complete".

(6. 17) *Example*. There exists a symmetric generalized uniform space $(X,\mathcal{U})$ and there exists a filter $\mathcal{J}$ on X such that $\mathcal{J}$ is weakly Cauchy with respect to $\mathcal{U}$, but $\mathcal{J}$ is not Cauchy with respect to $\mathcal{U}$.

Proof. Let $(X,\mathcal{I})$ be any connected T_1 topological space with at least two distinct points. By Corollary (1. 14) there exists a symmetric generalized proximity δ on X such that $\mathcal{I}(\delta)=\mathcal{I}$. Since $\mathcal{I}$ is T_1, it follows by Theorem (2. 15) that δ is separated. Let $\mathcal{U}_1(\delta)$ be the uniformity on X constructed in Theorem (2. 23). Consider the filter $\mathcal{J}=\{X\}$ on X. As shown in the proof of Theorem (6. 15), $\mathcal{J}$ is weakly Cauchy with respect to $\mathcal{U}_1(\delta)$. Let x_1 and x_2 be any two distinct points in X. Consider U_{x_1,x_2}. Since δ is separated, $x_1\bar{\delta}x_2$; so that $U_{x_1,x_2}\in\mathcal{U}_1(\delta)$. Hence $\mathcal{J}$ is not Cauchy with respect to $\mathcal{U}_1(\delta)$.

(6. 18) **Theorem.** *If $(X,\mathcal{U})$ has an open base, then every convergent filter on X relative to $\mathcal{I}(\mathcal{U})$ is a Cauchy filter.*

PROOF. Let $U\in\mathcal{U}$. Since $\mathcal{U}$ has an open base, there exists $U_1\in\mathcal{U}$ such that $U\supset U_1$ and U_1 is open in the product topology on $(X\times X)$. Suppose $\mathcal{J}$ is a filter on X which converges to x_0. Since U_1 is open, there exists an open set $A\in\mathcal{N}(x_0)$ the neighborhood system of x_0, such that $(A\times A)\subset U_1$. But $A\in\mathcal{J}$. Hence $\mathcal{J}$ is Cauchy with respect to $\mathcal{U}$.

(6. 19) **Theorem.** *Every convergent filter on $(X,\mathcal{U})$ is weakly Cauchy.*

PROOF. Let $\mathcal{J}$ be a filter on X. Suppose $\mathcal{J}$ converges to $x_0\in X$ relative to $\mathcal{I}(\mathcal{U})$. Let $U\in\mathcal{U}$. By Corollary (2. 12) $U(x_0)\in\mathcal{N}(x_0)$, the neighborhood system of x_0. Hence $U(x_0)\in\mathcal{J}$.

(6. 20) *Remark*. Take the usual base $\mathcal{B}$ for the usual uniformity $\mathcal{U}$ on the reals R and remove (except $(0,0)$) the points that lie on the line $y=-x$ from each element in $\mathcal{B}$. $\mathcal{B}^*$, the collection of the modified elements of $\mathcal{B}$, is a base for a symmetric generalized uniformity $\mathcal{U}^*$ on R such that $\mathcal{I}(\mathcal{U}^*)$ is the real topology, and the neighborhood system of 0 is a convergent filter which is not Cauchy.

(6. 21) Let $(X,\mathcal{U})$ be a correct uniform space. A filter $\mathcal{J}$ on X is Cauchy with respect to $\mathcal{U}$ if it is weakly Cauchy with respect to $\mathcal{U}$.

Proof. Suppose $\mathcal{J}$ is weakly Cauchy with respect to $\mathcal{U}$. Let $U\in\mathcal{U}$. There exist $V\in\mathcal{U}$ such that $(V\circ V)\subset U$; and there exists $x_0\in X$ such that $V[x_0]\in\mathcal{J}$. Let $(a,b)\in(V[x_0]\times V[x_0])$. Then $a\in V[x_0]$ and $b\in V[x_0]$; consequently, $(a,b)\in(V\circ V)\subset U$.

(6. 22) *Remark.* The following facts about intrafilters are easily established. Let $\mathscr{J}$ be an infrafilter and let $\mathscr{J}_1$ be a Cauchy filter in the correct uniform space $(X, \mathscr{U})$. Let $\mathscr{U}(\mathscr{J}_1)=\{U[F] | F\in\mathscr{J}_1$ and $U\in\mathscr{U}\}$. (A) $\mathscr{U}(\mathscr{J}_1)$ is an infrafilter that is contained in $\mathscr{J}_1$. (B) $\mathscr{J}_1$ is an infrafilter iff for every $A\in\mathscr{J}_1$, there exists a $B\in\mathscr{J}_1$ and a $U\in\mathscr{U}$ such that $A\supset U[B]$. (C) $\mathscr{N}(x)$, the neighborhood system of the point x, is an infrafilter in $(X, \mathscr{U})$. (D) $\mathscr{J}$ has an open base. (E) For every U, V in $\mathscr{U}$ there exists a $W\in\mathscr{U}$ such that if $F\in\mathscr{J}$ and $(F\times F)\subset W$, then $(F\times F)\subset(U\cap V)$. (F) If $(X_a, \mathscr{U}_a)$ is a dense subspace of the correct uniform space $(X_b, \mathscr{U}_b)$ and if $\mathscr{J}_0$ is an infrafilter in $(X_b, \mathscr{U}_b)$ then $\mathscr{B}=\{F\cap X_a | F\in\mathscr{J}_0$ and $U\in\mathscr{U}_b$ is a base for an infrafilter $\mathscr{J}_0^*$ in $(X_a, \mathscr{U}_a)$.

(6. 23) *Remark.* Note that in the proof of Theorem (2. 6) we actually only used the following weak form of (M. 4):

(M. 4)* For every $x\in X$ and U, V in $\mathscr{U}$ there is a $W\in\mathscr{U}$ such that $W[x]\subset\subset U[x]\cap V[x]$.

Let X be a non-void set. Let $\mathscr{U}$ be a non-void subset of $P(X\times X)$. $\mathscr{U}$ is a *semicorrect uniformity* on X iff $\mathscr{U}$ satisfies (M. 2), (M. 3), (M. 4)*, (M. 7), and (M. 8). By our above statement we have that if $(X, \mathscr{U})$ is a semicorrect uniform space, then the function $g: P(X)\to P(X)$ defined by $x\in g(A)$ iff $U[x]\cap A\neq\emptyset$ for all $U\in\mathscr{U}$ is a Kuratowski closure function. By a straightforward computation it is possible to show that if $(X_a, \mathscr{U}_a)$ is a dense subspace of the semicorrect uniform space $(X_b, \mathscr{U}_b)$, then $(X_a, \mathscr{U}_a)$ is a separated correct uniform space iff $(X_b, \mathscr{U}_b)$ is a separated correct uniform space. Also, it is easy to show that $\mathscr{B}$, a subset of $P(X\times X)$, is a base for some semicorrect uniformity on X iff $\mathscr{B}$ satisfies (M. 2), (M. 3), (M. 4)*, and (M. 7). We prove the former.

Fix $A\subset X_b$ and U, V in $\mathscr{U}_b$. There exists a U_1, V_1 in $\mathscr{U}_b$, such that $U_1=U_1^{-1}$, $V_1=V_1^{-1}$, $U_1\circ U_1\circ U_1\subset U$ and $V_1\circ V_1\circ V_1\subset V$. We let $B=U_1[A]\cap V_1[A]\cap X_a$ and choose to B considering the correctness of $\mathscr{U}_a$ a set $W\in\mathscr{U}_b$, such that $W_1=W_1^{-1}$ and $W_1^*[B]\subset U_1^*[B]\cap V_1^*[B]$. Where $(*)$ denotes here the trace taken in $X_a\times X_a$. If in addition $W\in\mathscr{U}_b$ $W_1\in\mathscr{U}_b$ $W\circ W\circ W\subset W_1$ and $y\in W[A]$, then there exists an $a\in A$ such that $(a, y)\in W$, and there exists a b such that $b\in W[a]\cap U_1[a]\cap V_1[a]\cap X_a$, since X_a is dense. In this case $b\in B$ and there exists also an x, such that $x\in W[y]\cap\cap U_1[y]\cap V_1[y]\cap X_a$. Thus $(b, x)\in W_1^*$, $x\in W_1^*[B]$, therefore $x\in U_1[B]\cap V_1[B]$ and $y\in U_1[x]\subset U_1[U_1[U_1[A]]]\subset U[A]$. Similarly, $y\in V[A]$.

(6. 24) **Theorem.** *Let $(X, \mathscr{U})$ be a separated, correct uniform space. The following are equivalent:*

(a) *$(X, \mathscr{U})$ is Δ-complete.*

(b) *Every infrafilter is a neighborhood system of some point.*

(c) *Every Cauchy filter on $(X, \mathscr{U})$ converges.*

(d) *$(X, \mathscr{U})$ is complete.*

PROOF. (a)$\to$(b). The neighborhood system of the point x will be denoted by $\mathscr{N}(x)$. Suppose there exists at least one infrafilter on X which is not a neighborhood system of a point in X. Let X_b be the family of all infrafilters on X. Let X_a be the family of all neighborhood systems of points in X. It is clear that $X_a\subset X_b$. For each $U\in\mathscr{U}$ we let $\overline{U}=\{(P_1, P_2) | (F\times F)\subset U$ for some $F\in P_1\cap P_2\}$. Let $\mathscr{B}=\{\overline{U} | U\in\mathscr{U}\}$.

We show that $\mathscr{B}$ is a base for a semicorrect uniformity $\mathscr{U}_b$ on X_b. By Remark (6. 23) it is sufficient to show that $\mathscr{B}$ satisfies (M. 2), (M. 3), (M. 4)*, and (M. 7).

(M. 2): Suppose $P_1, P_2 \in X_b$, $P_1 \neq P_2$ and $(P_1, P_2) \in \overline{U}$ for every $\overline{U} \in \mathscr{B}$. Then for every $U \in \mathscr{U}$ there exists $F \in (P_1 \cap P_2)$ such that $(F \times F) \subset U$. Hence $P_3 = (P_1 \cap P_2)$ is a Cauchy filter; so that since $P_3 \subset P_1$ and $P_3 \subset P_2$ we have by Definition (6. 7) that $P_1 = P_2 = P_3$ which is a contradiction.

(M. 3): Since $U = U^{-1}$ for every $U \in \mathscr{U}$, it is clear that $\overline{U}^{-1} = \overline{U}$ for every $\overline{U} \in \mathscr{B}$.

(M. 4)*: Let $P \in X_b$ and let $\overline{U}$, $\overline{V}$ be in $\mathscr{B}$. By Remark (6. 22E) there exists $W \in \mathscr{B}$ such that for all $F \in P$ if $(F \times F) \subset W$ then $(F \times F) \subset (U \cap V)$. We claim that $\overline{W}[P] \subset \overline{U}[P] \cap \overline{V}[P]$. For suppose $P_1 \in \overline{W}[P]$. Then $(P, P_1) \in \overline{W}$; so that there exists $F \in (P \cap P_1)$ such that $(F \times F) \subset W$ and hence such that $(F \times F) \subset (U \cap V)$. Consequently, $P_1 \in \overline{U}[P] \cap \overline{V}[P]$.

(M. 7): Suppose $\overline{U} \in \mathscr{B}$. There exists $V \in \mathscr{U}$ such that $V \subset U$ and $(V \circ V) \subset U$. We claim that $\overline{V} \circ \overline{V} \subset \overline{U}$. Suppose $(P_1, P_2) \in \overline{V}$ and $(P_2, P_3) \in \overline{V}$. Then there exists $F \in (P_1 \cap P_2)$ such that $(F \times F) \subset V$ and there exists $G \in (P_2 \cap P_3)$ such that $(G \times G) \subset \subset V$. But this implies for some $E \in (P_1 \cap P_3)$ that $(E \times E) \subset U$. Hence $(P_1, P_3) \in \overline{U}$. (Let $E = G \cup F$.)

Consequently, $\mathscr{U}_b = \{\overline{U} | \overline{U} = \overline{U}^{-1}$ and $\overline{U} \supset \overline{V}$ for some $\overline{V} \in \mathscr{B}\}$ is a semicorrect uniformity on X_b.

Consider the mapping $h: X \to X_b$ defined by $h(x) = \mathscr{N}(x)$. Since $(X, \mathscr{U})$ is separated, $\mathscr{I}(\mathscr{U})$ is T_0; so that h is 1—1. Clearly, h is onto X_a. Let $\overline{U} \in \mathscr{B}$. There exists open $V \in \mathscr{U}$ such that $V \subset U$ and $V \circ V \subset U$. Suppose $(x, y) \in V$. Then by a straightforward calculation it can be shown that if $F = (V[x] \cap V[y])$, then $(F \times F) \subset U$ and $F \in \mathscr{N}(x) \cap \mathscr{N}(y)$; so that $(\mathscr{N}(x), \mathscr{N}(y)) \in \overline{U}$. Conversely, suppose $(\mathscr{N}(x), \mathscr{N}(y)) \in \overline{U}$. Then it is immediate that $(x, y) \in U$. Hence we have that $(X, \mathscr{U})$ is uniformly isomorphic to $(X_a, \mathscr{U}_a)$ where $\mathscr{U}_a$ is the relativization of $\mathscr{U}_b$ to X_a.

Suppose P_1 is any point in X_b. Let $\overline{U}$ be any element of $\mathscr{B}$. $(P_1, P_1) \in \overline{U}$; so that by Remark (6. 22D) there exists an open set $F \in P_1$ such that $(F \times F) \subset U$. Let $x_0 \in F$. Then $F \in \mathscr{N}(x_0)$; so that $(P_1, \mathscr{N}(x_0)) \in \overline{U}$. Hence X_a is dense in X_b. Consequently, by Remark (6. 23) we have that $(X_b, \mathscr{U}_b)$ is a separated correct uniform space.

Thus we see that if there exists at least one infrafilter which is not the neighborhood system of some point in X, then it is possible to construct a separated correct uniform space $(X_b, \mathscr{U}_b)$ such that $(X, \mathscr{U})$ is uniformly isomorphic to a dense subspace $(X_a, \mathscr{U}_a)$ of $(X_b, \mathscr{U}_b)$ and $X_a \neq X_b$. Consequently, $(X, \mathscr{U})$ is not Δ-complete.

Proof (b) → (a). Suppose not. Then $(X, \mathscr{U})$ is uniformly isomorphic to a dense subspace $(X_a, \mathscr{U}_a)$ of $(X_b, \mathscr{U}_b)$ and $X_a \neq X_b$. Suppose $P \in (X_b - X_a)$. Let $\mathscr{J} = \mathscr{N}(P)$. Since $\mathscr{J}$ is an infrafilter, by Remark (6. 22F) it induces in $(X_a, \mathscr{U}_a)$ an infrafilter $\mathscr{J}^*$ But by hypothesis $\mathscr{J}^* = \mathscr{N}(P_1)$ for some point $P_1 \in X_a$. Hence $P \in \bigcap \{F | F \in \mathscr{J}\}$ and $P_1 \in \bigcap \{F | F \in \mathscr{J}\}$. But since $\mathscr{J}$ is Cauchy, this means that $(P, P_1 \in \overline{U}$ for every $\overline{U} \in \mathscr{U}_b$, and since $(X_b, \mathscr{U}_b)$ is separated this is a contradiction.

Proof (b) → (c). Let $\mathscr{J}$ be a Cauchy filter in $(X, \mathscr{U})$. By Remark (6. 22A) $\mathscr{J}$ contains an infrafilter $\mathscr{J}_1$ in $(X, \mathscr{U})$. But by hypothesis $\mathscr{J}_1 = \mathscr{N}(x_0)$ for some $x_0 \in X$. Hence $\mathscr{J}$ converges to x_0.

Proof (c) → (b). Let $\mathscr{J}$ be an infrafilter in $(X, \mathscr{U})$. By hypothesis $\mathscr{J} \supset \mathscr{N}(x_0)$ for some $x_0 \in X$. But $\mathscr{N}(x_0)$ is a Cauchy filter in $(X, \mathscr{U})$. Hence $\mathscr{J} = \mathscr{N}(x_0)$.

Proof $(c)\to(d)$. Let $\mathcal{J}$ be a weakly Cauchy filter with respect to $\mathcal{U}$. By Theorem (6. 21) $\mathcal{J}$ is a Cauchy filter with respect to $\mathcal{U}$. But then $\mathcal{J}$ is convergent and hence has a cluster point.

Proof $(d)\to(c)$. Let $\mathcal{J}$ be a Cauchy filter with respect to $\mathcal{U}$. By Theorem (6. 14) $\mathcal{J}$ is a weakly Cauchy filter with respect to $\mathcal{U}$ and hence has a cluster point. By Remark (6. 12G) $\mathcal{J}$ is convergent.

This completes the proof of Theorem (6. 24) which is essentially the same as that given by Efremovič, Mordkovič, and Sandberg in [8].

(6. 25) **Theorem.** *If* $(X, \mathcal{U})$ *is totally bounded, then every ultrafilter on* X *is a weakly Cauchy filter.*

Proof. Let $\mathcal{J}$ be an ultrafilter in $(X, \mathcal{U})$. Let $V\in\mathcal{U}$. There exist $x_1, \dots, x_n$ in X such that $X = V(x_1)\cup\cdots\cup V(x_n)$. But then since $X\in\mathcal{J}$, we have by Remark (6. 12C) that for some m where $1\leqq m\leqq n$ $V(x_m)\in\mathcal{J}$.

(6. 26) **Theorem.** $(X, \mathcal{U})$ *is complete and totally bounded iff* $(X, \mathcal{T}(\mathcal{U}))$ *is compact.*

Proof. Assume $(X, \mathcal{T}(\mathcal{U}))$ is compact. Let $U\in\mathcal{U}$. Consider the family $\{U[x]|x\in X\}$. By Corollary (2. 12) for each $x\in X$ $x\in[U[x]]^0$; consequently, for each $x\in X$ there exists an open set O_x such that $x\in O_x\subset U[x]$. Hence since $\mathcal{T}(\mathcal{U})$ is compact, there exists $x_1, \dots, x_n$ such that $X = U(x_1)\cup\cdots\cup U(x_n)$; so that $(X, \mathcal{U})$ is totally bounded. Let $\mathcal{J}$ be a weakly Cauchy filter. By Remark (6. 12E) since $\mathcal{T}(\mathcal{U})$ is compact, $\mathcal{J}$ has a cluster point; so that $(X, \mathcal{U})$ is complete.

Conveisely, let $\mathcal{J}$ be an ultrafilter on X. Since $(X, \mathcal{U})$ is totally bounded we have by Theorem (6. 25) that $\mathcal{J}$ is weakly Cauchy. But since $(X, \mathcal{U})$ is complete, $\mathcal{J}$ has a cluster point; so that by Remark (6. 12D) $\mathcal{J}$ is convergent. Consequently, by Remark (6. 12F) $\mathcal{T}(\mathcal{U})$ is compact. This proof is essentially the same as that given by Naimpally and Murdeshwar for Theorem (4. 14) in [35].

(6. 27) *Corollary*. Let $(X, \mathcal{U})$ be a separated, correct uniform space. Then $(X, \mathcal{T}(\mathcal{U}))$ is compact iff $(X, \mathcal{U})$ is totally bounded, and every infrafilter on X is a neighborhood system of some point in X.

Proof. This is an immediate consequence of Theorem (6. 24) and Theorem (6. 26).

(6. 28) *Corollary*. Every closed subspace $(Y, \mathcal{V})$ of a complete space $(X, \mathcal{U})$ is a complete space.

Proof. Let $(Y, \mathcal{V})$ be a closed subspace of $(X, \mathcal{U})$. Let $\mathcal{J}_1$ be any weakly Cauchy filter on Y relative to $\mathcal{V}$. $\mathcal{J}_1$ can be considered as a filter base for a filter $\mathcal{J}_1^*$ on X. It is clear that $\mathcal{J}_1^*$ is weakly Cauchy on X, relative to $\mathcal{U}$ and hence has a cluster point $x_0\in X$. But then x_0 is a cluster point of $\mathcal{J}_1$; so that x_0 is an accumulation point of Y. Since Y is closed, $x_0\in Y$. Hence $(Y, \mathcal{V})$ is complete.

(6. 29) *Definition.* Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{U}$ be any structure on X which generates a topology $\mathcal{T}(\mathcal{U})$ on X. Then $\mathcal{U}$ is *compatible with* $(X, \mathcal{T})$ iff $\mathcal{T}(\mathcal{U})=\mathcal{T}$.

(6. 30) **Theorem.** *A symmetric topological space $(X, \mathcal{I})$ is compact iff it is complete with respect to every compatible symmetric generalized uniformity $\mathcal{U}$ on X.*

Proof. Let $(X, \mathcal{U})$ be compatible with $(X, \mathcal{I})$. By Theorem (6. 26) $(X, \mathcal{U})$ is complete.

Conversely, we know by Corollary (1. 14) that there exists a symmetric generalized proximity δ on X such that $\mathcal{I}(\delta)=\mathcal{I}$. Let $\mathcal{U}_1(\delta)$ be the symmetrix generalized uniformity on X constructed in Theorem (2. 23). We know that $\mathcal{I}(\mathcal{U}_1(\delta))=\mathcal{I}$; so that by hypothesis $\mathcal{U}_1(\delta)$ is complete. But by Remark (3. 5) $\mathcal{U}_1(\delta)$ is totally bounded. Hence by Theorem (6. 26) $\mathcal{I}$ is compact.

(6. 31) *Remark.* Note the analogy between Theorem (6. 30) and the theorem of Niemytzki and Tychonoff which states that a metrizable topological space is compact iff it is complete in every compatible metric (cf. [37]). Also, recall the theorem of Doss which states that a completely regular topological space $(X, \mathcal{I})$ is compact iff it is complete with respect to every compatible uniformity $\mathcal{U}$ on X (cf. [6]).

(6. 32) **Theorem.** *$(X, \mathcal{U})$ is totally bounded iff every filter on X is contained in a weakly Cauchy filter.*

Proof. Suppose $(X, \mathcal{U})$ is totally bounded. Let $\mathcal{J}$ be a filter on X. By Remark (6. 12B) $\mathcal{J}$ is contained in an ultrafilter $\mathcal{J}_1$ which by Theorem (6. 25) is weakly Cauchy.

Conversely, suppose every filter on X is contained in a weakly Cauchy filter. Let $U\in\mathcal{U}$. For every finite subset $E\subset X$, assume that $U[E]\neq X$; so that $(X-U[E])\neq\emptyset$. The family $\{X-U[E]|E$ a finite subset of $X\}$ is easily shown to be a base for a filter, which by hypothesis is contained in a weakly Cauchy filter $\mathcal{J}$. For some point $x_0\in XU[x_0]\in\mathcal{J}$. On the other hand, since $\{x_0\}$ is a finite set $(X-U[x_0])\in\mathcal{J}$. But since $U[x_0]\cap(X-U[x_0])=\emptyset$; we have that $\emptyset\in\mathcal{J}$ which is a contradiction. This proof is essentially the same as that given by Sieber and Pervin for Theorem (1. 1) in [39].

(6. 33) **Theorem.** *If $(X, \mathcal{U})$ is a totally bounded, dense subspace of $(X_a, \mathcal{U}_a)$, and if every element of every weakly Cauchy filter on X_a has a non-void interior (relative to $\mathcal{I}(\mathcal{U}_a)$), and if every weakly Cauchy filter (relative $\mathcal{U}$) on X has a cluster point in X_a, then $(X_a, \mathcal{U}_a)$ is complete.*

PROOF. Let $\mathcal{J}$ be weakly Cauchy on X_a such that for every $F\in\mathcal{J}$ $F^0\neq\emptyset$. Since X is dense X_a, $(F\cap X)\neq\emptyset$ for every $F\in\mathcal{J}$. Let $\mathcal{B}=\{F\cap X|F\in\mathcal{J}\}$. Clearly, $\mathcal{B}$ is a base for a filter $\mathcal{J}_1$ on X which by Theorem (6. 32) is contained in a weakly Cauchy filter $\mathcal{J}_2$ on X. But by hypothesis $\mathcal{J}_2$ has a cluster point $x_0\in X_a$. Let $U\in\mathcal{U}_a$ and let $F\in\mathcal{J}$. Then $U[x_0]\cap(F\cap X)\neq\emptyset$; so that $U[x_0]\cap F\neq\emptyset$. Hence x_0 is a cluster point for $\mathcal{J}$, and $(X_a, \mathcal{U}_a)$ is complete.

(6. 34) **Theorem.** *If $(X, \mathcal{U})$ is separated, and Δ-complete, then every weakly Cauchy filter on X is the neighborhood system of some point in X.*

PROOF. The neighborhood system of the point x will be denoted by $\mathcal{N}(x)$. Suppose there exists at least one weakly Cauchy filter on X which is not a neighborhood system of a point in X. Let X_b be the family of all weakly Cauchy filters on X. Let X_a be the family of all neighborhood systems of points in X. It is clear that $X_a \subset X_b$. To construct the uniformity $\mathscr{U}_b$ on X_b in the proper way we assign to each filter P in the set X_b a point $x_P \in X$ in the following way: $x_P = x_1$ if $P = \mathscr{N}(x_1)$, and x_P is any point in X if $P \neq \mathscr{N}(x)$ for every $x \in X$. For each $U \in \mathscr{U}$ we let $\overline{U} = \{(P_1, P_2) | (x_{P_1}, x_{P_2}) \in U\}$. Let $\mathscr{B}$ equal $\{\overline{U} | U \in \mathscr{U}\}$. We show that $\mathscr{B}$ is a base for a symmetric generalized uniformity $\mathscr{U}_b$ on X_b. By Theorem (2. 22) it is sufficient to show that $\mathscr{B}$ satisfies (M. 1), (M. 3), (M. 4), and (M. 6).

(M. 1): Let $\overline{U} \in \mathscr{B}$. Since $(x_P, x_P) \in U$ for every $P \in X_b$ we have that $(P, P) \in \overline{U}$ for every $P \in X_b$.

(M. 3): Since $U = U^{-1}$ for every $U \in \mathscr{U}$, we have that $\overline{U}^{-1} = \overline{U}$ for every $\overline{U} \in \mathscr{B}$.

(M. 4): Let $A^* \subset X_b$ and let $\overline{U}$, $\overline{V}$ be in $\mathscr{B}$. Let $A = \{x_p | P \in A^*\}$ .There exists by (M. 4) a $W \in \mathscr{U}$ such that $W[A] \subset U[A] \cap V[A]$. Let $P_1 \in \overline{W}[A^*]$. Then $(P_a, P_1) \in \overline{W}$ for some $P_a \in A^*$; so that $(x_{P_a}, x_{P_1}) \in W$. Consequently, $x_{P_1} \in W[A]$; so that $x_{P_1} \in U[A] \cap V[A]$. But this means that there exists $x_{P_r} \in A$ and $x_{P_s} \in A$ such that $(x_{P_r}, x_{P_1}) \in U$ and $(x_{P_s}, x_{P_1}) \in V$; so that $P_1 \in \overline{U}[A^*] \cap \overline{V}[A^*]$. Hence, there exists a $\overline{W} \in \mathscr{B}$ such that $\overline{W}[A^*] \subset \overline{U}[A^*] \cap \overline{V}[A^*]$.

(M. 6): Let $A^* \subset X_b$ and $B^* \subset X_b$. Let $\overline{U} \in \mathscr{B}$ and let $\overline{V} \in \mathscr{B}$. Suppose $\overline{V}[A^*] \cap \cap B^* \neq \emptyset$.

Let $A = \{x_P | P \in A^*\}$. Let $B = \{x_P | P \in B^*\}$. Let $P_c \in \overline{V}[A^*] \cap B^*$. Then $P_c \in \overline{V}[A^*]$ and $P_c \in B^*$; so that for some $P_a \in A^*$ we have $(P_a, P_c) \in \overline{V}$ and hence $(x_{P_a}, x_{P_c}) \in V$. Consequently, since $\overline{V}$ is any element in $\mathscr{B}$ $V[A] \cap B \neq \emptyset$ for all $V \in \mathscr{U}$. But by (M. 6) there exists a $W \in \mathscr{U}$ and an element $x_{P_b} \in B$ such that $W[x_{P_b}] \subset U[A]$. Let $P_1 \in \overline{W}[P_b]$. Then $(x_{P_b}, x_{P_1}) \in W$ and $x_{P_1} \in W[x_{P_b}]$; so that there exists $x_{P_d} \in A$ such that $(x_{P_d}, x_{P_1}) \in U$ or equivalently, $(P_d, P_1) \in \overline{U}$ and $P_1 \in \overline{U}[A^*]$. Hence $\overline{W}[P_b] \subset \subset \overline{U}[A^*]$. Consequently, $\mathscr{U}_b = \{\overline{U} | \overline{U} = \overline{U}^{-1}$ and $\overline{U} \supset \overline{V}$ for some $\overline{V} \in \mathscr{B}\}$ is a symmetric generalized uniformity on X_b.

Consider the mapping $h: X \to X_b$ defined by $h(x) = \mathscr{N}(x)$. Since $(X, \mathscr{U})$ is separated, $\mathscr{I}(\mathscr{U})$ is T_0; so that h is 1—1. Clearly h is onto X_a. Let $U \in \mathscr{U}$. Let $(x, y) \in U$. Then $(\mathscr{N}(x), \mathscr{N}(y)) \in \overline{U}$. Conversely, suppose $(\mathscr{N}(x), \mathscr{N}(y)) \in \overline{U}$. Then $(x, y) \in U$. Hence we have that $(X, \mathscr{U})$ is uniformly isomorphic to $(X_a, \mathscr{U}_a)$ where $\mathscr{U}_a$ is the relativization of $\mathscr{U}_b$ to X_a.

Suppose P_1 is any point in X_b. Let $\overline{U}$ be any element of $\mathscr{B}$. $(P_1, P_1) \in \overline{U}$; so that $(x_{P_1}, x_{P_1}) \in U$. Hence $(P_1, \mathscr{N}(x_{P_1})) \in \overline{U}$; so that X_a is dense in X_b.

Thus we see that if there exists at least one weakly Cauchy filter which is not the neighborhood system of some point in X, then it is possible to construct a symmetric generalized uniform space $(X_b, \mathscr{U}_b)$ such that $(X, \mathscr{U})$ is uniformly isomorphic to a dense subspace $(X_a, \mathscr{U}_a)$ of $(X_b, \mathscr{U}_b)$ and $X_a \neq X_b$. Consequently, $(X, \mathscr{U})$ is not Δ-complete.

(6. 35) *Remark.* If $(X_b, \mathscr{U}_b)$ as constructed in the proof of Theorem (6. 34) is complete, then $(X, \mathscr{U})$ is complete. For suppose $\mathscr{J}$ is weakly Cauchy on X. Let $\mathscr{J}^* = \{h(F) | F \in \mathscr{J}\}$. $\mathscr{J}^*$ is clearly a base for a filter $\mathscr{J}_1^* \in X_b$. $\mathscr{J}_1^*$ is weakly Cauchy with respect to $\mathscr{U}_b$. Thus $\mathscr{J}_1^*$ has a cluster point $P_1 \in X_b$. P_1 is also a cluster point for $\mathscr{J}^*$. Let $F \in \mathscr{J}$ and let $F^* = h(F)$. Let $U \in \mathscr{U}$. Then there exists $\mathscr{N}(x_1) \in \overline{U}[P_1] \cap F^*$; so that $x_1 \in U[x_{P_1}] \cap F$. Consequently, x_{P_1} is a cluster point for $\mathscr{J}_1$ and $(X, \mathscr{U})$ is complete. Thus we see that the construction used in the proof of Theorem (6. 34) does not yield a completion for $(X, \mathscr{U})$.

(6. 36) **Theorem.** *Every separated correct uniform space has a unique completion.*

Proof. We show that $(X_b, \mathscr{U}_b)$ as constructed in the proof of Theorem (6. 24) is complete. The proof is essentially the same as that given by Efremovič, Mordkovič, and Sandberg in [8]. Let $\overline{U} \in \mathscr{B}$. Let $\bar{\mathscr{J}}$ be an infrafilter in $(X_b, \mathscr{U}_b)$. By Remark (6. 22F) $\bar{\mathscr{J}}$ induces in X_a the infrafilter $\mathscr{J}^*$ in $(X_a, \mathscr{U}_a)$ which is the natural image under the map h (as defined in the proof of theorem (6. 24)) of filter $\mathscr{J}$ in $(X, \mathscr{U})$. We now show that $\mathscr{J}$, which of course is an element of X_b, is a cluster point for $\bar{\mathscr{J}}$. By Remark (6. 22D) there exists an open $\overline{G} \in \bar{\mathscr{J}}$ such that $(\overline{G} \times \overline{G}) \subset \overline{U}$. Let $G^* = \overline{G} \cap X_a$. Let $G = h^{-1}(G^*)$. It is clear that G is open in X, $G \in \mathscr{J}$ and $(G \times G) \subset U$. Hence for every $x \in G$ we have that $\mathscr{N}(x) \in \overline{U}[\mathscr{J}]$; so that $G^* \subset \overline{U}[\mathscr{J}]$. But by Remark (6. 22F) every element of $\bar{\mathscr{J}}$ meets G^*. Hence $\mathscr{J}$ is a cluster point for $\bar{\mathscr{J}}$. But by Remark (6. 12G) $\bar{\mathscr{J}} \supset \mathscr{N}(\mathscr{J})$; so that since $\bar{\mathscr{J}}$ is an infrafilter $\bar{\mathscr{J}} = \mathscr{N}(\mathscr{J})$. Consequently, by Theorem (6. 24) $(X_b, \mathscr{U}_b)$ is complete.

That the completion is unique is shown in a straightforward manner.

(6. 37) *Remark*. The existence of a completion for more general types of symmetric generalized uniform spaces is an open question.

VII

Symmetric generalized topological groups

In this chapter we introduce the concept of a symmetric generalized topological group and show its relationship to symmetric generalized uniform spaces. We then extend regular Haar measure to locally compact Hausdorff symmetric generalized topological groups, and show that this measure will be essentially unique if we require that the groups be compact.

Throughout this chapter $(G, \cdot)$ will denote a group with identity ε. $\mathscr{J}$ will denote a topology on G, and $\mathscr{N}$ will denote an open base at ε. If A and B are subsets of G, $AB = \{ab \mid a \in A, b \in B\}$ and $A^{-1} = \{a^{-1} \mid a \in A\}$.

(7. 1) *Definition.* $(G, \cdot, \mathscr{J})$ is a symmetric generalized topological group iff the following axioms are satisfied:

(A. 1) For every $x \in G$ $\{xN \mid N \in \mathscr{N}\}$ and $\{Nx \mid N \in \mathscr{N}\}$ is an open base at x.

(A. 2) For every $N \in \mathscr{N}$ $N = N^{-1}$.

(7. 2) *Remark*. If we require that the mapping $f:(x, y) \to xy$ of $(G \times G)$ onto G be continuous in each variable separately, then $\{xN \mid N \in \mathscr{N}\}$ and $\{Nx \mid N \in \mathscr{N}\}$ are bases at x for every $x \in G$. If we require that the mapping $g: x \to x^{-1}$ of G onto G be continuous, then for every $N \in \mathscr{N}$ $N^{-1} \in \mathscr{N}$. This latter fact implies that for every $N \in \mathscr{N}$ $(N \cap N^{-1}) \in \mathscr{N}$. But $(N \cap N^{-1})^{-1} = (N \cap N^{-1})$.

(7. 3) *Remark*. It is easily shown that if F is closed. P is open, and A is an arbitrary subset of G and if x is an arbitrary point in G, then xF, F^{-1} are closed and xP, P^{-1}, and AP are open subsets of G where $(G, \cdot, \mathscr{J})$ is a symmetric generalized topological group.

(7. 4) **Theorem.** *If* $(G, \cdot, \mathscr{I})$ *is a symmetric generalized topological group, then* $\bar{A} = \bigcap \{AN \mid N \in \mathscr{N}\}$.

PROOF. Let $y \in \bar{A}$ and $N \in \mathscr{N}$. Then $yN^{-1} \cap A \neq \varnothing$; so that $y \in AN$. Conversely, suppose $y \in AN$ for every $N \in \mathscr{N}$. Then $y \in AN^{-1}$ for every $N \in \mathscr{N}$; consequently, $yN \cap A \neq \varnothing$ for every $N \in \mathscr{N}$; so that $y \in \bar{A}$.

(7. 5) **Theorem.** *Let* $(G, \cdot, \mathscr{I})$ *be a symmetric generalized topological group. For each* $N \in \mathscr{N}$ *let* $U_N = \{(x, y) \mid x^{-1}y \in N\}$. *Let* $\mathscr{B} = \{U_N \mid N \in \mathscr{N}\}$. *Then* $\mathscr{B}$ *is a base for a symmetric generalized uniformity* $\mathscr{U}(G)$, *on G such that* $\mathscr{I}(\mathscr{U}(G)) = \mathscr{I}$. *Note that even* (M. 5) *is true.*

(7. 6) *For every A, B of G and* $N \in \mathscr{N}$, *if* $AM \cap B \neq \varnothing$ *for all* $M \in \mathscr{N}$, *then there exists* $b \in B$ *and there exists a* $W \in \mathscr{N}$ *such that* $bW \subset AN$.

PROOF. $AN = \bigcup \{xN \mid x \in A\}$; so that by (A. 1) AN is open. But by hypothesis there exists $b \in AN \cap B$. Since AN is open, b is an interior point of AN; consequently, by (A. 1) there exists $W \in \mathscr{N}$ such that $bW \subset AN$.

PROOF *of Theorem* (7. 5). Clearly, to show $\mathscr{B}$ is a base for some symmetric generalized uniformity $\mathscr{U}$ on G it is sufficient to show that for every $N \in \mathscr{N}$ and for all subsets A, B of G, if $U_M[A] \cap B \neq \varnothing$ for every $M \in \mathscr{N}$, then there exists $b \in B$ and there exists $W \in \mathscr{N}$ such that $U_W[b] \subset U_N[A]$. But since we have that $U_N[A] = \bigcup \{xN \mid x \in A\} = AN$ for all $N \in \mathscr{N}$ and for each subset A of G, this is an immediate consequence of Lemma (7. 6). It is clear that $\mathscr{I}(\mathscr{U}(G)) = \mathscr{I}$.

(7. 7) *Corollary*. If $(G, \cdot, \mathscr{I})$ is a symmetric generalized topological group and $\mathscr{N}$ has a least element, say N_0, then $N_0^2 \subset N$ for every $N \in \mathscr{N}$.

Proof. Clearly, for every $U \in \mathscr{U}(G)$ we have that $U_{N_0} \subset U$. Consequently, by Lemma (2. 32) $U_{N_0} \circ U_{N_0} \subset U$ for every $U \in \mathscr{U}(G)$. Hence if $(x, y) \in U_{N_0}$, and $(y, z) \in U_{N_0}$, then $(x, z) \in U_N$ for every $N \in \mathscr{N}$. That is to say for every $N \in \mathscr{N}$ if $x^{-1}y \in N_0$ and $y^{-1}z \in N_0$, then $x^{-1}z \in N$. Let $p \in N_0$ and $q \in N_0$. Then p^{-1} is in N_0; so that $p^{-1}\varepsilon$ is in N_0 and $\varepsilon^{-1}q$ is in N_0. Hence $pq \in N$. Thus $N_0 N_0 \subset N$.

(7. 8) **Theorem.** *If* $(G, \cdot, \mathscr{I})$ *is a locally compact symmetric generalized topological group, then* $\mathscr{U}(G)$ *is complete.*

PROOF. Let $\mathscr{J}$ be any filter in G that is weakly Cauchy with respect to $\mathscr{U}(G)$. Since $(G, \cdot, \mathscr{I})$ is locally compact, there exists a compact neighborhood $N \in \mathscr{N}$, and since $\mathscr{J}$ is weakly Cauchy with respect to $\mathscr{U}(G)$, there exists an $x_0 \in G$ such that $U_N[x_0] = x_0 N \in \mathscr{J}$. By (A. 1) it is easily shown that $x_0 N$ is compact. We now let $\mathscr{B} = \{E \mid E = F \cap x_0 N$ for some $F \in \mathscr{J}\}$. It is easily shown that $\mathscr{B}$ is a base for a filter $\mathscr{J}_1$ in $x_0 N$; but since $x_0 N$ is compact, $\mathscr{J}_1$ has a cluster point $x_1 \in x_0 N$; which clearly is a cluster point for $\mathscr{J}$. Hence $(G, \mathscr{U}(G))$ is complete.

(7. 9) **Theorem.** *If* $(G, \cdot, \mathscr{I})$ *is a locally compact,* T_2 *symmetric generalized topological group, then* $(G, \cdot, \mathscr{I})$ *is a (symmetric) topological group.*

PROOF. This non trivial result is due to R. ELLIS. Cf. Problem B page 41 in [13].

Appendix I

Symmetric topological spaces

Let X be a non-void set with power set $P(X)$. Let I be an index set. A family $\{N_a|a\in I\}$ of functions from X into $P(X)$ which assign to each $x\in X$ a subset $N_a(x)\subset X$ is an *indexed system of neighborhoods for X which defines $\mathcal{J}$* iff the following conditions are satisfied:

(i) For each $x\in X$, $x\in \bigcap\{N_a(x)|a\in I\}$.

(ii) To each pair $a\in I$, $b\in I$, there corresponds at least one $c\in I$, such that for all $x\in X$, $N_c(x)\subset N_a(x)\cap N_b(x)$.

(iii) $O\in\mathcal{J}$ iff for each $x\in O$, there is an $a\in I$, such that $N_a(x)\subset O$.

(iv) Given $a\in I$, $x\in X$, and $y\in N_a(x)$, there is a $b\in I$, such that $N_b(y)\subset N_a(x)$.

(v) For each $a\in I$, $x\in N_a(y)$ implies $y\in N_a(x)$.

For each $a\in I$ we let $N^*=\{(x,y)|y\in N_a(x)\}$. If A and B are subsets of X, A is said to be *separated from B* by an open set C if $A\subset C$ and $B\cap C=\varnothing$.

In [5] A. S. DAVIS essentially shows that the following statements for a topological space $(X,\mathcal{J})$ are equivalent:

(i) $\mathcal{J}$ is symmetric.

(ii) Closed sets are separated from the points that they exclude.

(iii) Every open set contains the closure of each of its points.

(iv) There exists an indexed system of neighborhoods which defines $\mathcal{J}$ and has the additional property that for every $a\in I$ N^* is open in the product topology on $(X\times X)$ derived from $\mathcal{J}$.

(v) For all $x\in X$, $y\in Y$, $\bar{x}\cap\bar{y}\neq\varnothing$ implies $\bar{x}=\bar{y}$.

(vi) $\mathcal{J}$ is isomorphic (lattice-theoretically) to the topology of a T_1 space.

K. MORITA in [30], [31], and [32] has defined the concepts of "completeness" and "completion" for a symmetric topological space, and he has shown that every such space has a "completion".

Appendix II

Precorrect uniform spaces

Let $\mathcal{U}$ be a subset of $P(X\times X)$. Let (M. 6)* be the axiom: for every $A\in P(X)$ and for every $U\in\mathcal{U}$ there exist V, W in $\mathcal{U}$ such that $(W\circ V)(A)\subset U[A]$.

The following theorem is proved by Mordkovič in [28].

(A) *Theorem.* Suppose for each $U\in\mathcal{U}$ $U=U^{-1}$. Define a relation $\delta(\mathcal{U})$ on $P(X)$ by $A\delta(\mathcal{U})B$ iff $U[A]\cap B\neq\varnothing$ for all $U\in\mathcal{U}$. Then $\delta(\mathcal{U})$ satisfies (P. 1), (P. 2), (P. 3), (P. 4), and (P. 5)′ iff $\mathcal{U}$ satisfies (M. 1), (M. 4), and (M. 6)*.

(B) *Definition.* $\mathcal{U}$ is a *precorrect uniformity* on X iff $\mathcal{U}$ satisfies (M. 1), (M. 3), (M. 4), (M. 6)*, and (M. 8). If $\mathcal{U}$ is a precorrect uniformity on X, then $(X,\mathcal{U})$ is called a *precorrect uniform space.*

It is easily shown that (M. 7) implies (M. 6)* and that (M. 6)* implies (M. 6). Consequently, every symmetric uniform space is a precorrect uniform space, and every precorrect uniform space is a symmetric generalized uniform space.

The following theorems about precorrect uniform spaces are established in virtually the same way as the corresponding theorems about symmetric generalized uniform spaces if we substitute Theorem (A) for Theorem (2. 2).

Let (X, δ) be a proximity space, and let $\pi(\delta)$ be a proximity class of precorrect uniformities on X.

(C) *Theorem.* $\mathscr{B}$, a subset of $P(X \times X)$, is a base for some symmetric generalized uniformity on X iff $\mathscr{B}$ satisfies (M. 1), (M. 3), (M. 4), and (M. 6)*.

(D) *Theorem.* $\pi(\delta)$ contains one and only one totally bounded symmetric uniformity.

(E) *Theorem.* $\pi(\delta)$ contains a maximum and minimum element.

(F) *Theorem.* If δ is the usual proximity for the reals X, then $\pi(\delta)$ contains at least two distinct totally bounded precorrect uniformities that have an open base.

(G) *Theorem.* If $(X, \mathscr{T})$ is connected completely regular topological space, then there exists a totally bounded precorrect uniformity $\mathscr{U}$ on X with an open base such that $\mathscr{T}(\mathscr{U}) = \mathscr{T}$ and every filter in X is weakly Cauchy with respect to $\mathscr{U}$.

(H) *Theorem.* A precorrect uniform space is compact iff it is complete and totally bounded.

(I) *Theorem.* A completely regular topological space $(X, \mathscr{T})$ is compact iff it is complete with respect to every compatible precorrect uniformity on X.

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