Bol quasigroups

By D. A. ROBINSON (Atlanta, Ga.)

§ 1. Introduction

A binary system $(G, \cdot)$ is said to be a quasigroup if and only if whenever any two of the ordered triple x, y, z are given as elements in G the third is uniquely determined in G so that $x \cdot y = z$. Recall also that a loop is a quasigroup which has a two-sided identity element. We shall say that a quasigroup $(G, \cdot)$ is a Bol quasigroup if and only if

$$(xy \cdot z)y = x(yz \cdot y) \tag{1}$$

for all $x, y, z \in G$. A Bol loop is a loop which is also a Bol quasigroup.

Bol loops as well as other generalizations of Moufang loops have received renewed attention in recent years (see, for instance, the author [6], [7], [8] and F. FENYVES [3], [4]). It would seem natural to investigate Bol quasigroups. However, it is the purpose of this paper to show that the theories of Bol quasigroups and Bol loops are, in a sense, coextensive. Specifically, a loop-theoretic construction of Bol quasigroups is given (see Theorem 1) and, moreover, it is shown that this construction accounts for all Bol quasigroups (see Theorem 2). It is interesting to note that our development stems from the fact (see Lemma 1; see also CHOUDHURY [2]) that a Bol quasigroup necessarily possesses a left identity element.

It is well-known (see BRUCK [1]) that any quasigroup is an isotope of some loop. The reader who is familiar with isotopy theory will realize that our main results (Theorems 1 and 2) indicate that a Bol quasigroup can be realized as a very special isotope of a Bol loop. We have largely ignored the isotopy aspects in this paper since they seem neither to enhance the results nor to facilitate the proofs. It should be noted that this paper is based for the most part on a heretofore unpublished portion of the author's Ph. D. dissertation [5].

§ 2. Preliminary lemmas

We commence with a few basic results concerning Bol quasigroups.

Lemma 1. *If $(G, \cdot)$ is a Bol quasigroup, then $(G, \cdot)$ has a unique left identity element.*

PROOF. Let x be a fixed element in G. Since $(G, \cdot)$ is a quasigroup, there exists an element $e \in G$ so that $ex = x$. Now for each $w \in G$ there exist elements $y, z \in G$

so that $yx=w$ and $xz=y$. Then $ew = e\cdot yx = e(xz\cdot x) = (ex\cdot z)x = xz\cdot x = yx = w$. So, indeed, e is a left identity element for $(G, \cdot)$. The fact that e is the only left identity element for $(G, \cdot)$ is an obvious consequence of $(G, \cdot)$ being a quasigroup.

Henceforth, let e denote the unique left identity element for a Bol quasigroup $(G, \cdot)$ and, for each $x\in G$, let x^λ and x^ϱ be those unique elements in G so that $x^\lambda x = xx^\varrho = e$. This brings us to

Lemma 2. *If $(G, \cdot)$ is a Bol quasigroup, then*

(i) *$x^\varrho = x^\lambda$ for all $x\in G$,*

(ii) *$(G, \cdot)$ satisfies the right inverse property.*

Proof. For $x\in G$ we have $x(x^\varrho x\cdot x^\varrho) = (xx^\varrho\cdot x)x^\varrho = ex\cdot x^\varrho = xx^\varrho$. So $x^\varrho x\cdot x^\varrho = x^\varrho$ and we have $x^\varrho x = e$. Thus, $x^\varrho = x^\lambda$ for all $x\in G$ and (i) holds. Also, for $x, y\in G$, we have $(xy\cdot y^\varrho)y = x(yy^\varrho\cdot y) = x\cdot ey = xy$. So $xy\cdot y^\varrho = x$ for all $x, y\in G$ and (ii) holds.

If x is any element of a Bol quasigroup $(G, \cdot)$, we shall, in view of Lemma 2, let $x^{-1} = x^\varrho = x^\lambda$.

Lemma 3. *Let $(G, \cdot)$ be a Bol quasigroup and let g be any fixed element in G. Then $(G, \circ)$ is a Bol loop where $x\circ y = xg\cdot y$ for all $x, y\in G$.*

Proof. It is easy to see that $(G, \circ)$ is a quasigroup. If e is the left identity element of the Bol quasigroup $(G, \cdot)$ (see Lemma 1), then it also easily follows that $e\cdot g^{-1} = g^{-1}$ is a two-sided identity element for $(G, \circ)$. Hence, $(G, \circ)$ is a loop. Note also that, for all $x, y, z\in G$, we have

$$[((xg^{-1})\circ y)\circ z]\circ y = \{[(xg^{-1}\cdot g)y]g\cdot z\}g\cdot y = [(xy\cdot g)z]g\cdot y = [(xy)(gz\cdot g)]y$$

and

$$(xg^{-1})\circ[(y\circ z)\circ y] = (xg^{-1}\cdot g)[(yg\cdot z)g\cdot y] = x\{[y(gz\cdot g)]y\} = [(xy)(gz\cdot g)]y.$$

(Note that we have made repeated use of (1) and Lemma 2.) Hence, we have

$$[((xg^{-1})\circ y)\circ z]\circ y = (xg^{-1})\circ[(y\circ z)\circ y]$$

for all $x, y, z\in G$ and it follows that $(G, \circ)$ is a Bol loop.

§ 3. Main results

We shall now examine explicitly the connection between Bol quasigroups and Bol loops. Such a relation has already been suggested by Lemma 3.

Theorem 1. *Let $(G, \circ)$ be a Bol loop and let θ be an automorphism of $(G, \circ)$ such that $\theta^2 = I$ where I is the identity mapping on G. If $x\cdot y = x\theta\circ y$ for all $x, y\in G$, then $(G, \cdot)$ is a Bol quasigroup.*

PROOF. It is easily established that $(G, \cdot)$ is a quasigroup. Now for $x, y, z \in G$ we have

$$(xy \cdot z)y = [(x\theta \circ y)\theta \circ z]\theta \circ y = [(x\theta \circ y)\theta^2 \circ z\theta] \circ y = [(x\theta \circ y) \circ z\theta] \circ y =$$

$$= x\theta \circ [(y \circ z\theta) \circ y] = x\theta \circ [(y\theta^2 \circ z\theta) \circ y] = x\theta \circ [(y\theta \circ z)\theta \circ y] = x(yz \cdot y).$$

Hence, the quasigroup $(G, \cdot)$ is a Bol quasigroup.

Now we proceed to show that all Bol quasigroups can be constructed from Bol loops in precisely the manner described in Theorem 1. Specifically, we have

Theorem 2. *If $(G, \cdot)$ is a Bol quasigroup, then there is a binary operation $\circ$ for G so that $(G, \circ)$ is a Bol loop, $(G, \circ)$ has an automorphism θ with the property that $\theta^2 = I$, and $x \cdot y = x\theta \circ y$ for all $x, y \in G$.*

PROOF. Let e be the left identity element for the Bol quasigroup $(G, \cdot)$ (see Lemma 1). For all $x, y \in G$ define $x \circ y$ by $x \circ y = xe \cdot y$. By Lemma 3 with $g = e$ we see that $(G, \circ)$ is a Bol loop and its two-sided identity element is e. Define θ by $x\theta = x \cdot e$ for all $x \in G$. Since $(G, \cdot)$ is a quasigroup, it is clear that θ is a one-to-one mapping of G onto G. Furthermore, since $(G, \cdot)$ satisfies the right inverse property (see Lemma 2) and $e^{-1} = e$, we have $x\theta^2 = xe \cdot e = x$ for all $x \in G$. Hence, $\theta^2 = I$. Also note that $(x \circ y)\theta = (xe \cdot y)e = x(ey \cdot e) = x \cdot ye = (xe \cdot e)(ye) = x\theta \circ y\theta$ for all $x, y \in G$. Hence, θ is an automorphism of $(G, \circ)$ and $\theta^2 = I$. Finally observe that $x\theta \circ y = (xe \cdot e)y = x\theta^2 \cdot y = x \cdot y$ for all $x, y \in G$.

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