Arcs and bases

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The main result contained in this paper is the following: A separable infinite dimensional Banach space has a monotone Schauder basis ([1], p. 69) if and only if it has a completely monotone arc whose linear extension is dense in the space.

The statement that an arc α, in a normed linear space, is completely monotone means there is a homeomorphism h of $[0, 1]$ onto α such that $\bigcup_{n=0}^{\infty} D_n$ is a monotone collection where for each nonnegative integer n,

$$D_n = \{2h(t/2^n) - h(t+1)/2^n) - h((t-1)/2^n): \quad 1 \leq t \leq 2^n - 1\}.$$

To establish the above mentioned result suppose that $\{\varphi_i\}_{i=1}^{\infty}$ is a normalized, $(\|\varphi_i\| = 1,\ i = 1, 2, \dots)$ monotone Schauder basis for a separable Banach space X. Define a sequence of functions $\{f_i\}_{i=1}^{\infty}$ each from $[0, 1]$ into X as follows:

$$f_1(x) = x\varphi_1 \quad \text{if} \quad x \in [0, 1],$$

$$f_{n+1}(x) = 2^{n-1}[x - j/2^{n-1}]\big[f_n((j+1)/2^{n-1}) - f_n(j/2^{n-1}) + (\sqrt{2})^{1-n}\varphi_{2^{n-1}+j+1}\big] + \\ + f_n(j/2^{n-1}) \quad \text{if} \quad x \in [0, 1] \cap [2j/2^n, (2j+1)/2^n]$$

and

$$f_{n+1}(x) = 2^{n-1}[x - (2j+1)/2^n]\big[f_n((j+1)/2^{n-1}) - f_n(j/2^{n-1}) - (\sqrt{2})^{1-n}\varphi_{2^{n-1}+j+1}\big] + \\ + f_{n+1}((2j+1)/2^n) \quad \text{if} \quad x \in [0, 1] \cap ((2j+1)/2^n, (2j+2)/2^n).$$

A straightforward computation yields $\|f_{n+1}(x) - f_n(x)\| \leq (\sqrt{2})^{1-n}$. We have therefore a sequence of continuous functions from $[0, 1]$ into X converging uniformly to $f: f(x) = \lim_{n\to\infty} f_n(x)$.

If u and v are numbers in $[0, 1]$ then $f(u) = \sum_{i=1}^{\infty} a_i\varphi_i$ and $f(v) = \sum_{i=1}^{\infty} b_i\varphi_i$ and since $\{\varphi_i\}_{i=1}^{\infty}$ is a Schauder basis, this representation is unique. From the construction of f, it is easily observed that, there is a positive integer n such that $a_n \neq b_n$ and hence $f(u) \neq f(v)$. Therefore f is a homeomorphism.

It is also clear from the construction that the linear extension of $\alpha = f([0, 1])$ is dense in X.

To show that α is completely monotone observe that

$$f_{n+1}(t/2^{n-1}) = f_n(t/2^{n-1}) = f(t/2^{n-1})$$

and that

$$f((2j+1)/2^n) = \frac{1}{2}\left[f((j+1)/2^{n-1}) - f(j/2^{n-1}) - (\sqrt{2})^{1-n}\varphi_{2^{n-1}+j+1}\right] + f(j/2^{n-1}).$$

It follows then that

$$2f((2j+1)/2^n) - f((2j+2)/2^n) - f(2j/2^n) = (\sqrt{2})^{1-n}\varphi_{2^{n-1}+j+1}$$

and therefore α is completely monotone.

We have constructed then a completely monotone arc α such that the linear extension of α is dense in X.

Suppose now that an infinite dimensional Banach space X has a completely monotone arc α such that the extension of α is dense in X. Define

$$\varphi_{2^{n-1}-j+1} = 2h((2j+1)/2^n) - h((2j+2)/2^n) - h(2j/2^n) \quad \text{for} \quad n=0, 1, 2, \ldots;$$
$$0 \leqq j \leqq 2^{n-1}-1$$

where h is such a homeomorphism as given in the definition of completely monotone. Note that for each t, $0 \leqq t \leqq 2^n$ that $h(t/2^n)$ is in the linear extension of $\{\varphi_i : i=1, 2, \ldots\}$ and that $H = \{h(t/2^n) : 0 \leqq t \leqq 2^n;\ n=0, 1, 2, \ldots\}$ is dense in α. If x is in the linear extension of H then x has a unique representation $\sum_{i=1}^{\infty} a_i\varphi_i$ since $\{\varphi_i : i=1, 2, \ldots\}$ is a monotone set. Moreover if $\lim_{j\to\infty} x_j = x$ where for each positive integer j, x_j is in the linear extension of H then $x = \sum_{i=1}^{\infty} a_i\varphi_i$ where for each positive integer i $a_i = \lim_{j\to\infty} a_{ij}$ and $x_j = \sum_{i=1}^{\infty} a_{ij}\varphi_i$. This representation of x is unique also.

Hence $\{\varphi_i\}_{i=1}^{\infty}$ is a Schauder basis for X, which completes the argument.

If k is any homeomorphism of [0, 1] into a normed linear space Y and $\varphi_1, \varphi_2, \ldots$ is a sequence in Y as defined above (replacing h by k) then it is easy to verify that $\varphi_1 = [1/(1-2^{-n})][k(1-2^{-n}) - \sum_{t=1}^{n} [1/(2^{n-t+1}-2^{-t+1})]\varphi_{2^t}$. Hence, noting that $\varphi_1 = k(1)$, we have that

$$\varphi_1 = \sum_{t=1}^{\infty} 2^{t-1}\varphi_{2^t} \quad \text{if} \quad \lim_{n\to\infty} 2^n[k(1) - k(1-2^{-n})] = 0.$$

Thus one sees that the character of h given in the definition of completely monotone is critical.

The following propositions are readily verified and the proofs are omitted.

1. If C is a collection of nonoverlapping chords [2, p. 5] of a completely monotone arc then C is a monotone collection.

2. If α is a completely monotone arc then the nonzero elements of α are linearly independent [3].

References

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(Received September 11, 1970.)