The space of bounded maps into a Banach space

By E. MAKAI, Jr. (Budapest)

Let D be a real B-space satisfying the following conditions: 1) D is strictly convex, 2) for every $v, v \in D^*$, $\|v\|=1$ the set $\{w, w \in D, \|w\| \leqq 1, v(w) = 1-\delta\}$ contains a u_1 for which
$\{w, w \in D, \|w-u_1\| \leqq r, v(w) = 1-\delta\} \subset \{w, w \in D, \|w\| \leqq 1, v(w) = 1-\delta\}$,
where $r/\delta \to \infty$ if $0<\delta<1$, $\delta \to 0$ and v is fixed, 3) D has no proper subspace isometrically isomorphic to D, 4) D is infinite dimensional. Let X be a realcompact space, i.e. completely regular and for every $p \in \beta X \setminus X$ there is an $f \in C(X)$ not extensible to p. $C^* = C^*(X, D)$ denotes the B-space of bounded continuous functions from X to D.

Theorem. *C^*, as a B-space, determines X and D. More accurately if for $C^*(X_i, D_i)$ (X_i, D_i belonging to the above classes) we have a linear isometry ψ of $C^*(X_1, D_1)$ onto $C^*(X_2, D_2)$, then there exist a homeomorphism $\varphi: X_2 \to X_1$ and a continuous map A from X_1 to the isometrical isomorphies of D_1 onto D_2 (taken in the strong operator topology) such that $(\psi f)(x_2) = A(\varphi(x_2)) \cdot f(\varphi(x_2))$.*

PROOF. 1. For later use we consider first the case $D=R$, and X is compact T_2 (condition 4 is not satisfied). We denote $C^*(X, R)$ by B.

$f \in B$ is an extremal point of the unit sphere if and only if $|f(x)|=1$ for every $x \in X$.

Let f_0 be an extremal point of the unit sphere. Let A be a continuous map from X to the isometrical isomorphies of R to itself.

After applying such an A to each element of B (like above, with φ=identity) we can suppose $f_0=1$. For $f \in B$ we consider $\|f-\lambda f_0\|$, where $\lambda \in R$. This is

$$\max(|\max f-\lambda|, |\min f-\lambda|).$$

Its minimum is attained for $\lambda = (\max f+\min f)/2$ and has value $(\max f-\min f)/2$. So we can determine $\max f$ and $\min f$.

Let $f_1, f_2 \in B$. We shall determine the convex hull $S(f_1, f_2)$ of the range $R(f_1, f_2)$ of the vector-valued function (f_1, f_2). This consists of those points $(x_1, x_2) \in R^2$ for which for every $(\lambda_1, \lambda_2) \in R^2$ we have $\min(\lambda_1 f_1+\lambda_2 f_2) \leqq \lambda_1 x_1+\lambda_2 x_2 \leqq \leqq \max(\lambda_1 f_1+\lambda_2 f_2)$.

For those elements $f \in B$ for which $\min f \geqq 0$ and $\max f \leqq 1$ we write $f_1 \sim f_2$ if $(x_1, x_2) \in S(f_1, f_2)$ implies that either $(x_1, x_2)=(0, 0)$ or $x_1>0$ and $x_2>0$. This condition is equivalent to the following: $(x_1, x_2) \in R(f_1, f_2)$ implies that either $(x_1, x_2)=(0, 0)$ or $x_1>0$ and $x_2>0$, i.e. $f_1^{-1}(0)=f_2^{-1}(0)$. Let the class of f by the equivalence relation $\sim$ be $\tilde{f}$. We say $\tilde{f}_1 < \tilde{f}_2$ if $g_1 \in \tilde{f}_1$, $g_2 \in \tilde{f}_2$, $(x_1, x_2) \in S(g_1, g_2)$, $x_1 > 0$ imply $x_2 > 0$. This means $f_1^{-1}(0) \supset f_2^{-1}(0)$.

Thus we obtained the lattice H (whose elements are the above $\tilde{f}$-s) dual to the lattice of the compact G_δ sets of X. By [3], p. 173 we obtain a homeomorphic copy of X, the points of which are the maximal ideals of H.

For $f \in B$ and $x \in X$ we determine $f(x)$. We can suppose $\max f - \min f \leqq 1$. Let $\min f \leqq c \leqq \max f$. Then $f = c - f_1 + f_2$ where $\min f_1 = 0$, $\min f_2 = 0$, $\max f_1 \leqq 1$, $\max f_2 \leqq 1$ and for every maximal ideal g of H one of $\tilde{f}_1$ and $\tilde{f}_2$ belongs to g (and this decomposition is unique). We say for those maximal ideals g that the value of f at the corresponding point is c, for which $\tilde{f}_1 \in g$ and $\tilde{f}_2 \in g$. If $\min f \leqq c \leqq \max f$ is not true, the value of f is not c in any point.

At the beginning we applied an operator A to each $f \in B$. This means that we determined X up to a homeomorphism and the functions $f \in B$ up to the application of some A. Thus for compact T_2 X_1 and X_2 and $D = R$ the linear isometries of $C^*(X_1, D)$ onto $C^*(X_2, D)$ are of the required form.

2. We turn to the proof of the theorem. Let C^* be given, at first we determine D and βX.

$f \in C^*$ is an extremal point of the unit sphere if and only if $\|f(x)\| = 1$ for every $x \in X$.

Let now $i: D \to C^*$ be any linear isometric embedding such that for every $d \in D$ id is an extremal point of the sphere with center in the origin and of radius $\|d\|$. Then for every $x \in X$ $\|(id)(x)\| = \|d\|$, thus $d \to (id)(x)$ is a linear isometry of D to a subspace of D, which by condition 3 has to be D. Thus the functions id can be carried over to the functions identically equal to d by the application of an operator A of the theorem (with $\varphi =$ identity) cf. [1] Theorem 6. 1. Up to the application of such an A we can suppose $(id)(x) = d$ for every $x \in X$.

If C^* is isometrically isomorphic to $C^*(X', D')$ (X', D' from the above classes) then there is a linear isometric embedding $i': D' \to C^*$ having the same property as i, so D' can be embedded in D. Similarly vice versa, so D and D' are isometrically isomorphic.

For $f \in C^*$ let $S(f)$ denote the intersection of the closed spheres of centers d, d is any element of D, and radii $\|f - id\|$. $S(f)$ is a weakly closed convex set. We prove that it is the weakly closed convex hull of the range of f. That is, if a $u' \in D$ does not belong to some closed half-space $\{w', v(w') \leqq c\}$ containing the range of f, then it does not belong to $S(f)$ either.

We note that $\{w, \|w - u_1\| \leqq r, v(w) = 1 - \delta\} \subset \{w, \|w\| \leqq 1, v(w) = 1 - \delta\}$ implies $K = \{w, \|w - u_1\| \leqq r, v(w) \leqq 1 - \delta\} \subset \{w, \|w\| \leqq 1, v(w) \leqq 1 - \delta\}$ if $0 < \delta < 1$. This can be proved for two-dimensional subspaces by elementary geometry and the statement follows. There exist r', u_1' such that $v(u_1') = c$ and the range of f belongs to $\{w', v(w') \leqq c, \|w' - u_1'\| \leqq r'\} = K'$. We define u, u_0' by the condition that $\{u_0', K', u_1', u'\}$ could be obtained from $\{0, K, u_1, u\}$ by magnification and translation. If δ is sufficiently small u does not belong to the unit sphere since $v(u) = (v(u') - c)r/r' + 1 - \delta > 1$. So u' does not belong to the closed sphere of center u_0' and radius r', while K' belongs to it.

Let $0 \neq v \in D^*$. $v(S(f))$ is the (closed, open or semi-open) interval determined by $\inf v(f)$, $\sup v(f)$. $vC^* = C^*(X)$. Like in 1 we determine βX and for every $f \in C^*$ $[v(f)]^\beta \in C^*(\beta X)$. For $f_i \in C^*$, $0 \leqq \inf v(f_i) \leqq \sup v(f_i) \leqq 1$, $i = 1, 2$, f_1 and f_2 are said to be equivalent if $v(f_1)\sim v(f_2)$. The equivalence class of f is denoted by $\tilde{f}^v$. These $\tilde{f}^v$-s are elements of the lattice H^v (ordered similarly as H). For $0\neq v_1, v_2$ the elements of H^{v_1} and H^{v_2} will be made to correspond so that the maximal ideals belonging to the same point $x\in\beta X$ correspond to each other. For linearly dependent v_1, v_2 this can be made easily. For linearly independent v_1, v_2 $\tilde{f}_1^{v_1}$ and $\tilde{f}_2^{v_2}$ will be made to correspond if for $g_1\in\tilde{f}_1^{v_1}$, $v_2(g_1)=0$, $g_2\in\tilde{f}_2^{v_2}$, $v_1(g_2)=0$ (such g_1, g_2 exist) $\tilde{g}_1^{v_1+v_2}=\tilde{g}_2^{v_1+v_2}$ (these are meaningful). The last equality is equivalent to $v_1(g_1)\sim v_2(g_2)$ (i.e. $\{[v_1(f_1)]^\beta\}^{-1}(0)=\{[v_2(f_2)]^\beta\}^{-1}(0)$). Thus we have determined βX and for every $f\in C^*$ and $0\neq v\in D^*$ $[v(f)]^\beta$.

We consider $D^{**}\supset D$ with the weak* topology. $f\in C^*$ can be extended to an $f^\beta:\beta X\to D^{**}$ continuous map, $\|f^\beta\|=\|f\|$ and for $v\in D^*$ $[v(f)]^\beta=v(f^\beta)$.

3. By condition 4 D contains $\{e_n\}_{n\in N}$ for which $\|e_n\|=1$ and for any $n, \lambda_0, \dots, \lambda_n$ $\|\lambda_0 e_0+\dots+\lambda_n e_n+e_{n+1}\|\geqq 1/2$. Let $R^+=\{x, x\in R, x\geqq 0\}$. We define $\varphi:R^+\to D$. φR^+ will belong to the unit sphere. For $x\in R^+$ $\varphi(x)=e_{[x]}\cdot(1-x+[x])+e_{[x+1]}\cdot(x-[x])$. φR^+ is closed in D. $\overline{\varphi R^+}$ denotes the closure of φR^+ in D^{**}. $(\overline{\varphi R^+}\setminus\varphi R^+)\cap D$ is closed in D. Let $x\in X$. If for $f\in C^*$ $f^\beta(\beta X)\subset\overline{\varphi R^+}$ and $f^\beta(x)\in\overline{\varphi R^+}\setminus\varphi R^+$ then for a closed-open neighbourhood U (in X) of x $f(U)\subset(\overline{\varphi R^+}\setminus\varphi R^+)\cap D$ so for a neighbourhood $V(=\overline{U})$ (in βX) of x $f^\beta(V)\subset\overline{\varphi R^+}\setminus\varphi R^+$. Let $x\in\beta X\setminus X$. Then there exists a continuous map $h:X\to R^+$ such that the extension $\bar{h}$ of h, $\bar{h}:\beta X\to R^+\cup\{\infty\}$ (one-point compactification) satisfies $\bar{h}(x)=\infty$. For $f=\varphi\circ h$ $f^\beta(\beta X)\subset\overline{\varphi R^+}$, $f^\beta(x)\in\overline{\varphi R^+}\setminus\varphi R^+$, but every neighbourhood of x contains an x' such that $f^\beta(x')\in\varphi R^+$.

Thus we have found the subspace X of βX. Thus we have determined X and found the value of every $f\in C^*$ at every $x\in X$ up to the application of an operator A of the theorem.

Remark. Let X be S-compact, where S is the unit sphere of D with the weak topology (i.e. let X be completely regular and for every $p\in\beta X\setminus X$ there is a continuous map $X\to S$ not extensible to p, cf. [4]). Let D have the above property 2. $C_w^*(X, D)$ denotes the B-space of the bounded weakly continuous functions from X to D. Let $C^*(X, D)\subset C^*\subset C_w^*(X, D)$ and let i map D into C^*, $(id)(x)=d$ for every $x\in X$. Then for the pair (i, C^*) a similar statement holds (with $A=$identity).

Acknowledgments. The author is indebted to T. Figiel who turned his attention to (a special case of) this problem. The author expresses his thanks to J. Gerlits. The idea and reason of considering realcompact spaces came from him, without which only compact spaces could be considered.

Rerefences

[1] M. Jerison, The space of bounded maps into a Banach space, *Ann. of Math.*, **52**, (1950), 309—327.
[2] L. Gillman—M. Jerison, Rings of continuous functions *Princeton* etc. 1960.
[3] G. Birkhoff, Lattice theory *New York*, 1948.
[4] H. Herrlich: 𝔈-kompakte Räume. *Math. Z.*, **96**, (1967), 228—255.

(Received January 20, 1971.)