On the orthonormal Franklin system

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1. Z. CIESIELSKI ([2] and [3]) investigated the properties of the orthonormal FRANKLIN system in many respect. His results show a far going similarity between HAAR and FRANKLIN systems. He also stated that most of the properties of Haar functions could be established also for Franklin system.

In this note we are going to give some statements about the convergence of the series of Fourier—Franklin coefficients in support of Ciesielski's opinion.

2. We shall use the following notations. The Haar functions are defined as follows:

$$\chi_1(t) \equiv 1 \quad \text{in} \quad [0, 1],$$

$$\chi_{2^{n+1}}(1) = -\sqrt{2^n},$$

$$\chi_{2^n+k} = \begin{cases} \sqrt{2^n} \text{ in } \left[\dfrac{2k-2}{2^{n+1}}, \dfrac{2k-1}{2^{n+1}}\right), \\ -\sqrt{2^n} \text{ in } \left[\dfrac{2k-1}{2^{n+1}}, \dfrac{2k}{2^{n+1}}\right), \\ 0 \text{ elsewhere in } [0, 1], \end{cases}$$

where $n=0, 1 \dots;\ k=1, 2, \dots, 2^n$.

Applying Schmidt's orthonormalization procedure to the Schauder functions:

$$\varphi_0(t) = 1, \quad \varphi_m(t) = \int_0^t \chi_{2^n+k}(\tau)\, d\tau \qquad t \in [0, 1],$$

where $m = 2^n+k$ ($n=0, 1, \dots;\ k=1, 2, \dots, 2^n$), we get the Franklin functions:

$$f_n(t) = \sum_{i=0}^{n} \lambda_{in} \varphi_i(t) \quad \text{with} \quad \lambda_{nn} > 0, \text{ for } n=0, 1, \dots,$$

where the triangular matrix (λ_{in}) is uniquely determined.

The n-th partial sum of the Fourier—Franklin series of a given $f(t)$ will be denoted by $S_n(f; t)$.

For a given $f(t) \in L_p[0, 1]$, $1 \leqq p \leqq \infty$ we put

$$\|f\|_p = \Big(\int_0^1 |f(t)|^p\, dt\Big)^{1/p}, \quad \|f\|_\infty = \|f\|.$$

Let the partitions $0=t_0<t_1<\dots<t_n=1$ of the interval $[0, 1]$ define as follows:

$$t_0=0 \quad \text{and} \quad t_1=1 \text{ for } n=1$$

and

$$t_i = \begin{cases} \dfrac{i}{2^{m+1}} & \text{for } i=0, 1, \dots, 2k \\ \dfrac{i-k}{2^m} & \text{for } i=2k+1, \dots, n \end{cases}$$

if $n = 2^m+k$ $(m=0, 1, \dots;\ k=1, 2, \dots, 2^m)$. Then the n-th degree polinomials of polygonals is defined by

$$\varphi = \varphi(n; t) = \frac{\xi_i-\xi_{i-1}}{t_i-t_{i-1}}(t-t_{i-1})+\xi_{i-1} \tag{1}$$

for $t_{i-1}\leqq t\leqq t_i$, $i=1, 2, \dots, n$. Now we can define the best approximation of a given function $f\in L_p[0, 1]$, by the n-th degree polynomials, in the following way:

$$E_n^{(p)}(f) = \inf_{\varphi}\|f-\varphi\|_p, \quad E_n^{\infty}(f) = E_n(f),$$

where the infimum is taken over all φ defined by (1).

The modulus of continuity of the first order in $L_p[0, 1]$ we define as follows:

$$\omega_1^{(p)}(\delta; f) = \sup_{0<h<\delta}\Big(\int_0^{1-h}|f(t+h)-f(t)|^p\,dt\Big)^{1/p}, \qquad 0<\delta<1;$$

and let

$$\omega(\delta; f) = \sup_{|x_1-x_2|\leqq\delta}|f(x_1)-f(x_2)|; \qquad x_1, x_2\in[0, 1].$$

3. We prove the following

Theorem. Let $\{\lambda_n\}$ be a positive monotonic sequence with $K\lambda_{2^n}\geqq\lambda_{2^{n+1}}\geqq K^{-1}\lambda_{2^n}$ $(K\geqq 1,\ n=1, 2, \dots)$ and let $f\in L_p[0, 1]$, $(1\leqq p<\infty)$, $0<\beta\leqq p$ and $a_n=(f, f_n)$. Then

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n^{\beta/2}}\left(E_n^{(p)}(f)\right)^{\beta}<\infty$$

implies

$$\sum_{n=1}^{\infty}\lambda_n|a_n|^{\beta}<\infty. \tag{2}$$

On the other hand if $0<\alpha<1$ and

$$\sum_{n=1}^{\infty}\frac{\lambda_n}{n^{\beta/2}}n^{-\alpha\beta}=\infty \tag{3}$$

then there exists a function $f\in \operatorname{Lip}\alpha$ such that for its Fourier—Franklin coefficients the series (2) is divergent.

4. This theorem implies some results of Z. Ciesielski ([3]). For instance the Corollary of Theorem 14 of [3] follows from our Theorem with $p=1$, $\lambda_n=1$, $\beta=1$; the first assertion of Theorem 17 of [3] with $\beta=1$, $\lambda_n=n^{\alpha}$, for $\alpha<\frac{1}{2}$ (using that $E_n^{(1)}(f)=O(1/n)$), the second assertion of Theorem 17 of [3] with $\lambda_n=1$, for $\beta>\frac{2}{3}$;

Theorem 18 of [3] with $\beta=2-\varepsilon$, $\lambda_n=1$; and Theorem 19 of [3] with $\beta=2-\varepsilon$. From our theorem we also obtain immediately some new results which are similar to the results published for Haar series by CIESIELSKI—MUSIELAK [1], GOLUBOV [4] and LEINDLER [5].

Corollary 1. Let $f(x)\in L_p[0,1]$ $(1\leqq p<\infty)$ and $\beta>0$, $\gamma>0$. Then

$$\sum_{n=1}^{\infty}\frac{\left[\omega_1^{(p)}\left(\frac{1}{n};f\right)\right]^{\beta}}{n^{\beta/2-\gamma}}<\infty$$

implies

$$\sum_{n=1}^{\infty} n^{\gamma}|a_n|^{\beta}<\infty.$$

If $\gamma\geqq 0$, $0<\alpha\leqq 1$ and $0<\beta\leqq 2(1+\gamma)/1+2\alpha$, then there exists a function $f(x)\in\operatorname{Lip}\alpha$, such that for its Fourier—Franklin coefficients:

$$\sum_{n=1}^{\infty} n^{\gamma}|a_n|^{\beta}=\infty.$$

Corollary 2. a) Let $p\geqq 1$ and let $f(x)$ be a function of bounded variation, in order p, that is

$$V_p(f)=\left\{\sup_B\sum_{i=1}^{n}|f(t_i)-f(t_{i-1})|^p\right\}^{1/p}<\infty,$$

for any $B(0=t_0<t_1<\dots<t_n=1)$; then for every

$$\beta<2p/(2+p)\quad\text{and}\quad\gamma<\frac{1}{p}-\frac{1}{2}$$

$$\sum_{m=1}^{\infty}|a_m|^{\beta}<\infty\quad\text{and}\quad\sum_{m=1}^{\infty}m^{\gamma}|a_m|<\infty$$

respectively.

b) For every $p\geqq 1$ there exists a function $f_p(x)$ of bounded variation in order p, such that for $\beta=2p/(2+p)$ and for $\gamma=(1/p)-\frac{1}{2}$

$$\sum_{m=1}^{\infty}|a_m|^{\beta}=\infty\quad\text{and}\quad\sum_{m=1}^{\infty}m^{\gamma}|a_m|=\infty,$$

respectively.

Corollary 3. Let $f(x)\in\operatorname{Lip}\alpha$ $(0<\alpha\leqq 1)$. If $p\geqq 1$, then

$$\text{(i)}\qquad\left\{\sum_{k=n+1}^{\infty}|a_k|^p\right\}^{1/p}=O\left(\frac{1}{n^{\alpha+1/2-1/p}}\right)\quad\text{for}\quad\alpha>\frac{1}{p}-\frac{1}{2};$$

$$\text{(ii)}\qquad\left\{\sum_{k=1}^{n}|a_k|^p\right\}^{1/p}=O(n^{1/p-1/2-\alpha})\quad\text{for}\quad\alpha<\frac{1}{p}-\frac{1}{2};$$

$$\text{(iii)}\qquad\left\{\sum_{k=1}^{n}|a_k|^p\right\}^{1/p}=O([\lg(n+1)]^{1/p})\quad\text{for}\quad\alpha=\frac{1}{p}-\frac{1}{2}.$$

5. We require the following lemmas:

Lemma 1. ([3], Corollary of Theorem 6). *Let $1 \leqq p \leqq \infty$, and let $\{a_n\}$ be an arbitrary sequence of real numbers. Then we have*

$$2^{-5}3^{-1/2}2^{m(1/2-1/p)}\left(\sum_{n=2^m+1}^{2^{m+1}}|a_n|^p\right)^{1/p} \leqq \left\|\sum_{n=2^m+1}^{2^{m+1}}|a_nf_n|\right\|_p \leqq$$

$$\leqq 2^5 3^{1/2} 2^{m(1/2-1/p)}\left(\sum_{n=2^m+1}^{2^{m+1}}|a_n|^p\right)^{1/p}$$

for $m \geqq 1$.

Lemma 2. ([3], Theorem 8). *If $f \in L_p[0,1]$ with $1 \leqq p < \infty$ or $f \in C[0,1]$, $(p=\infty)$, then we have*

$$E_n^{(p)}(f) \leqq \|f-S_n(f)\|_p \leqq 4E_n^{(p)}(f) \qquad (n \geqq 1).$$

Lemma 3. ([2], Corollary of Theorem 7.) *Let $f \in C[0,1]$, and let $0<\alpha<1$. Then the following conditions are equivalent*

(i) $$\|f-S_n(f)\| = O\left(\frac{1}{n^\alpha}\right) \quad \text{as} \quad n \to \infty,$$

(ii) $$\omega(\delta) = O(\delta^\alpha) \quad \text{as} \quad \delta \to 0_+,$$

(iii) $$|a_n(f)| = O\left(\frac{1}{n^{\alpha+1/2}}\right) \quad \text{as} \quad n \to \infty.$$

6. Proof of Theorem.

Our proof is similar to those of Ciesielski's theorems in [3]. From properties of $\{\lambda_n\}$, applying the Hölder's inequality we obtain

$$\sum_{n=2}^{\infty}\lambda_n|a_n|^\beta = \sum_{m=0}^{\infty}\sum_{n=2^m+1}^{2^{m+1}}\lambda_n|a_n|^\beta \leqq K\sum_{m=0}^{\infty}\lambda_{2^{m+1}}\sum_{n=2^m+1}^{2^{m+1}}|a_n|^\beta \leqq$$

$$\leqq K\sum_{m=0}^{\infty}\lambda_{2^{m+1}}2^{\frac{m(p-\beta)}{p}}\left\{\sum_{n=2^m+1}^{2^{m+1}}|a_n|^p\right\}^{\beta/p} =$$

$$= K\sum_{m=0}^{\infty}\lambda_{2^{m+1}}2^{m\left(1-\frac{\beta}{2}\right)}\left\{2^{m/2}\left[2^{-m}\sum_{n=2^m+1}^{2^{m+1}}|a_n|^p\right]^{1/p}\right\}^\beta.$$

Using Lemma 1 and Lemma 2 we have

$$2^{m/2}\left(2^{-m}\sum_{n=2^m+1}^{2^{m+1}}|a_n|^p\right)^{1/p} \leqq 2^5 3^{1/2}\|S_{2^{m+1}}(f)-S_{2^m}(f)\|_p \leqq 2^8 3^{1/2}E_{2^m}^{(p)}(f).$$

Hence *)

$$\sum_{n=2}^{\infty}\lambda_n|a_n|^\beta \leqq C\sum_{m=0}^{\infty}\lambda_{2^m}\cdot 2^{m\left(1-\frac{\beta}{2}\right)}[E_{2^m}^{(p)}(f)]^\beta$$

and from properties of $\{\lambda_n\}$ we have the first assertion of Theorem.

*) C is absolute constant, depending only on K.

In order to prove the second statement of Theorem we define

$$f(x)=\sum_{n=1}^{\infty} a_n f_n(x), \quad \text{where} \quad a_n=\frac{1}{n^{\alpha+1/2}}$$

A standard computation gives that $f(x)\in C[0,1]$ and by Lemma 3 we get that

$$f(x)\in \mathrm{Lip}\,\alpha.$$

Hence, by an easy computation, we obtain the second assertion of Theorem.
The proof is completed.

Rerefences

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