Decompositions of the Fredholm Operators

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We shall use the following notations:

H is a separable Hilbert space with infinite dimension.

$L(H)$ is the Banach algebra of all continuous linear transformations of H, $Gl(H)$ is the group of the inverible elements in $L(H)$.

$\text{Ker}\, A = \{\varphi | \varphi \in H,\ A\varphi = 0\}$ is the kernel of $A \in L(H)$ $\text{Coker}\, A = H/\text{Im}\, A$ where $\text{Im}\, A$ is the image of the operator A.

If $\text{Im}\, A$ is a closed subspace in H then $\text{Coker}\, A$ can be identified with $(\text{Im}\, A)^{\perp}$, the orthogonal complement of $\text{Im}\, A$.

An operator $A \in L(H)$ is called Fredholm operator if both $\dim \text{Ker}\, A$ and $\dim \text{Coker}\, A$ are finite. The difference

$$\dim \text{Ker}\, A - \dim \text{Coker}\, A$$

is called the index of the operator A.

The index of A and the space of all Fredholm operators will be denoted by Index A, and $\mathscr{F}$, respectively.

We mention that

$$\text{Index}\, A = \dim H/V - \dim H/A(V)$$

where $V \subset H$ is a subspace satisfying the next conditions:

$$\text{Ker}\, A \cap V = 0$$

$$\dim H/V < \infty \qquad \text{(see [1]).}$$

Let be

$$\mathscr{F}_k = \{A : A \in \mathscr{F},\ \text{Index}\, A = k).$$

It was proved that the space $\mathscr{F} \subset L(H)$ is an open set and its components are just the $\mathscr{F}_k$-s. Furthermore $\mathscr{F}_k \cdot \mathscr{F}_l \subset \mathscr{F}_{k+l}$ which can be expressed as

$$\text{Index}\, AB = \text{Index}\, A + \text{Index}\, B$$

(see [1], [2], [3]).

We deal with the question whether $\mathscr{F}_k \cdot \mathscr{F}_l$ and $\mathscr{F}_{k+l}$ can be identical, and we shall prove the following theorem

Theorem. *If $k \geqq 0$ or $l \leqq 0$ then $\mathscr{F}_k \cdot \mathscr{F}_l = \mathscr{F}_{k+l}$. If $k<0$ and $l>0$ then the equality $\mathscr{F}_k \cdot \mathscr{F}_l = \mathscr{F}_{k+l}$ does not hold. An operator $A \in \mathscr{F}_{k+l}$ can be decomposed in the form $A = A_1 A_2$ where $A_1 \in \mathscr{F}_k$, $A_2 \in \mathscr{F}_l$ if and only if*

$$\dim \operatorname{Ker} A \geqq l$$

or equivalently

$$\dim \operatorname{Coker} A \geqq -k.$$

Proof. Let us take an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of the Hilber space H, and define the following operators

$$T_k e_i = \begin{cases} e_{i-k} & \text{if } i>k \\ 0 & \text{if } i \leqq k \end{cases}$$

Obviously Index $T_k = k$ and if $k \geqq 0$, then $T_k T_{-k} = \mathrm{id}_H$. If $k \geqq 0$ or $l \leqq 0$ then an operator $A \in \mathscr{F}_{k+l}$ can be written as a product:

$$A = A_1 A_2 \quad \text{where} \quad A_1 \in \mathscr{F}_k, \quad A_2 \in \mathscr{F}_l.$$

The decomposition is the following:

$$A = T_k(T_{-k}A) \quad \text{or} \quad A = (AT_{-l})T_l$$

respectively.

If $k<0$ and $l>0$ a similar decomposition is not always possible. If $\dim \operatorname{Ker} A < l$, then a decomposition is surely impossible. Namely suppose $A = A_1 A_2$ where $A_1 \in \mathscr{F}_k$, $A_2 \in \mathscr{F}_l$. Then $\dim \operatorname{Ker} A_2 \geqq l$ and $l > \dim \operatorname{Ker} A = \dim \operatorname{Ker} A_1 A_2 \geqq \dim \operatorname{Ker} A_2 \geqq l$ which is a contradiction.

We obtain a necessary condition on the possibility of a decomposition. Clearly the operator T_{k+l} does not satisfy this condition, so in this case the equality $\mathscr{F}_k \mathscr{F}_l = \mathscr{F}_{k+l}$ does not hold.

The condition $\dim \operatorname{Ker} A \geqq l$ is equivalent to $\dim \operatorname{Coker} A \geqq -k$.

Indeed:

$$\dim \operatorname{Ker} A - l = \dim \operatorname{Coker} A + k \geqq 0$$

One of these conditions is, sufficient. Suppose then $\dim \operatorname{Ker} A \geqq l$. If $k+l \geqq 0$, we choose a subspace $V \subset H$ in the following manner:

V is the orthogonal complement of a subspace of $\operatorname{Ker} A$ having dimension $\dim \operatorname{Ker} A - (k+l)$. If $k+l < 0$ then V is defined to be the orthogonal complement of a subspace of dimension $\dim \operatorname{Ker} A - (k+l)$, which contains $\operatorname{Ker} A$. Since

$$\dim H/\operatorname{Im} A = \dim H/V$$

so there exists an operator $B \in Gl(H)$ which maps $\operatorname{Im} A$ onto V. For this B we have

$$\operatorname{Ker} BA = \operatorname{Ker} A, \quad \operatorname{Im} BA = V.$$

We decompose H as the direct sum of three orthogonal subspaces

$$H = (\operatorname{Ker} A \cap V^{\perp}) \oplus \{((\operatorname{Ker} A)^{\perp} \cap V^{\perp}) \oplus (\operatorname{Ker} A \cap V)\} \oplus ((\operatorname{Ker} A)^{\perp} \cap V) =$$

$$= H_1 \oplus H_2 \oplus H_3$$

Choose an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ in H such that $\{e_i\}_{i=1}^{n}$ is a basis for H_1', $\{e_i\}_{i=n+1}^{m}$ is a basis for H_2 and finally $\{e_i\}_{i=m+1}^{\infty}$ is a basis for H_3.

We define the action of a certain operator on this basis:

$$Ce_i = \begin{cases} (BA|_{(\mathrm{Ker}\,T)^\perp})^{-1} e_{i-k-l} & \text{if } e_i \notin \mathrm{Ker}\, A \\ e_i & \text{if } e_i \in \mathrm{Ker}\, A \end{cases}$$

The definition is correct, since the map

$$BA|_{(\mathrm{Ker}\,A)^\perp} : (\mathrm{Ker}\,A)^\perp \to V$$

is one to one.

Hence a basis of V is mapped onto a basis of $(\mathrm{Ker}\,A)^\perp$ by the inverse. Furthermore C is identity on $\mathrm{Ker}\,A$, so $C \in Gl(H)$. Let us compute the action of BAC on the given basis:

First if $e_i \in \mathrm{Ker}\,A$

$$BACe_i = BA(Ce_i) = BAe_i = 0.$$

Secondly if $e_i \notin \mathrm{Ker}\,A$ then

$$BACe_i = BA(Ce_i) = BA(BA|_{(\mathrm{Ker}\,A)^\perp})^{-1} e_{i-k-l} = e_{i-k-l} = T_k T_l e_i = T_{k+l-m} T_m e_i$$

if $i > m \geqq l$, where $m = \dim \mathrm{Ker}\,A$. Thus we obtain:

$$BAC = T_{k+l-m} T_m$$

On the other hand $T_{m-l} T_{l-m} = \mathrm{id}_H$ consequently

$$BAC = T_{k+l-m} T_{m-l} T_{l-m} T_m$$

Here $B, C \in Gl(H)$, so we rewrite this equality:

$$A = (B^{-1} T_{k+l-m} T_{m-l})(T_{l-m} T_m C^{-1})$$

which is the desired decomposition.

References

[1] M. F. Atiyah, Lectures on K-theory, *Cambridge (Mass.)*, 1965.

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