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Engel properties of group algebras I

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Abstract. We characterize 3-Engel group algebras, $p+1$-Engel group algebras of characteristic p, and show that the group of units of a 3-Engel group algebra is 3-Engel.

Introduction

Let G be a group and $\mathbb{F}$ a field. We define the Lie commutator $[x,y]$ of the elements x and y of the group algebra $\mathbb{F}G$ to be $xy-yx$, and the multiple Lie commutator $[x,y_1,y_2,\ldots,y_n]$ inductively to be $[[x,y_1,\ldots,y_{n-1}],y_n]$. We put $[x,y,y,\ldots,y]=[x,y,n]$, where the y occur n times, and say that the algebra $\mathbb{F}G$ is n-Engel if $[x,y,n]=0$ is an identity in $\mathbb{F}G$. We also say that $\mathbb{F}G$ is bounded Engel if it is n-Engel for some n. Recall SEHGAL's well-known result [4, Theorem V.6.1.]: $\mathbb{F}G$ is bounded Engel if and only if $\mathbb{F}G$ is commutative provided $\mathbb{F}$ is of characteristic 0; if and only if G is nilpotent containing a normal subgroup N such that the commutator subgroup N' and the factorgroup G/N are of p-power orders provided $\mathbb{F}$ is of prime characteristic p.

Let $\mathbb{F}G$ be n-Engel of prime characteristic p. In [3] RIPS and SHALEV proved that if $n<p$ then $\mathbb{F}G$ is commutative, and if $n=p$ then $|G'|\le p$. Extending these results, Theorem 1 and 2 determines 3 and $p+1$-Engel group algebras, respectively.

It is well-known that 3-Engel Lie algebras (even the not finitely generated ones) are nilpotent except the characteristic 2 and 5 cases. 3-Engel

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Lie algebras were studied by Traustason in [6], where an example of a 3-Engel non-nilpotent Lie algebra of characteristic 2 was given. We provide another example. Let $G = A \rtimes \langle b \rangle$, a semidirect product, where A is an infinite direct product of cyclic groups of order 4, and b is of order 2 acting by inversion on A. Then the group algebra of G over the prime field GF(2) is 3-Engel by Theorem 1, and not Lie nilpotent by Passi, Passman, Sehgal's characterization theorem [4].

Concerning the group commutator $(x, y) = x^{-1}y^{-1}xy$ we define the multiple commutators $(x, y_1, \dots, y_n)$ and (x, y, n), and the notions of an n-Engel and a bounded Engel group analogously as above. Group algebras with bounded Engel groups of units were described by Bovdi and Khripta [1].

In [5] Shalev proved that if A is an n-Engel associative algebra over a field of prime characteristic, then the group of units $U(A)$ is m-Engel for some m. Let $f(n)$ be the smallest possible such m. For group algebras it is easy to show $f(2) = 2$, and by means of Theorem 1 we also establish $f(3) = 3$ in Theorem 3. It would be interesting to assess $f(n)$ for greater n. Theorem 3 strongly suggests that we can tackle the problem of characterizing group algebras with 3-Engel group of units. This problem will be solved in the forthcoming second part of this paper.

In what follows, we assume G to be a group, and $\mathbb{F}$ to be a field of prime characteristic p. By $\gamma_k(G)$ we mean the kth term of the lower central series of G with $\gamma_1(G) = G$, by $\zeta(G)$ the center of G and by $\zeta(\mathbb{F}G)$ the center of $\mathbb{F}G$. For a subgroup $H \subseteq G$ we denote by $\mathcal{I}(H)$ the ideal in $\mathbb{F}G$ generated by all elements of form $h - 1$ with $h \in H$, and, with H finite, by $\widehat{H} \in \mathbb{F}G$ the sum of all elements of H. For a torsion element $a \in G$ we put $\widehat{\langle a \rangle} = \widehat{a}$. We shall use the following commutator identities frequently:

$$[x, yz] = [x, y]z + y[x, z], \quad [xy, z] = x[y, z] + [x, z]y,$$

in characteristic

$$p \quad [x, y, p] = [x, y^p], \quad [x, y] = yx((x, y) - 1), \quad (x, y) = 1 + x^{-1}y^{-1}[x, y],$$

$$(x, yz) = (x, z)(x, y)^z = (x, z)(x, y)(x, y, z),$$

$$(xy, z) = (x, z)^y(y, z) = (x, z)(x, z, y)(y, z).$$

Results

Theorem 1. *Let $\mathbb{F}$ be a field of prime characteristic p, G an arbitrary group. Then the group algebra $\mathbb{F}G$ is 3-Engel if and only if one of the following conditions holds:*

(i) *G is abelian;*

(ii) *$p = 2$ and G is nilpotent of class 2 with an elementary abelian commutator subgroup of order 2 or 4;*

(iii) *$p = 2$ and G is nilpotent of class 2 such that its commutator subgroup is an elementary abelian 2-group of either finite order greater than 4 or of infinite order, and there exists an abelian subgroup of index 2 in G;*

(iv) *$p = 3$ and G is nilpotent with a commutator subgroup of order 3.*

To establish Theorem 1 we need the following lemmas.

Lemma 1. *Let $\mathbb{F}G$ be a noncommutative $(p+1)$-Engel group algebra. Then $G/\zeta(G)$ is of exponent p.*

Proof. Pick $g, h \in G$ such that $(g,h) \neq 1$. Since $\mathbb{F}G$ is $(p+1)$-Engel,

$$\begin{aligned}[g,h,p+1] &= [g,h^p,h] = [h^p g((g,h^p)-1),h] \\ &= h^p\big(g[(g,h^p),h] + [g,h]((g,h^p)-1)\big) \\ &= h^p\big(gh(g,h^p)((g,h^p,h)-1) + hg((g,h)-1)((g,h^p)-1)\big) = 0.\end{aligned}$$

It follows

$$\begin{aligned}&(g,h)(g,h^p)((g,h^p,h)-1) + ((g,h)-1)((g,h^p)-1) \\ &\quad = (g,h)(g,h^p)(g,h^p,h) - (g,h) - (g,h^p) + 1 = 0.\end{aligned}$$

If $(g,h) = (g,h)(g,h^p)(g,h^p,h)$ then $(g,h^p) = 1$ as required. If not, then $(g,h) = (g,h^p)$ i.e. $(g,h) = (g,h)(g,h^{p-1})^h$, and therefore $(g,h^{p-1}) = 1$, which, since $G/\zeta(G)$ has to be of exponent either p or p^2, follows $(g,h) = 1$, a contradiction. □

Lemma 2. *Let $p = 2$ and let G be a nonabelian group with an abelian subgroup A of index 2 in G, and assume that $G/\zeta(G)$ is of exponent 2^m. Then $\mathbb{F}G$ is (2^m+1)-Engel.*

Proof. Pick some $b \in G$ such that $G/A \cong \langle bA\rangle$. Then every $y \in \mathbb{F}G$ can be written uniquely as $y = y_1 + y_2 b$ where $y_1, y_2 \in \mathbb{F}A$. Note that $b^2 \in \zeta(G) \subset A$, $by_1 = y_1^b b$, and $by_1 b = y_1^b b^2 \in \mathbb{F}A$ for any $y_1 \in \mathbb{F}A$. Clearly,

$$y^2 = (y_1 + y_2 b)^2 = y_1^2 + y_2 y_2^b b^2 + (y_1 + y_1^b) y_2 b,$$

$y_2 y_2^b b^2$ and $y_1 + y_1^b$ are central in $\mathbb{F}G$, $y^2 \equiv y_1^2 + (y_1 + y_1^b) y_2 b (\operatorname{mod} \zeta(\mathbb{F}G))$, and by induction it is easy to show

$$y^{2^k} \equiv y_1^{2^k} + (y_1 + y_1^b)^{2^k - 1} y_2 b (\operatorname{mod} \zeta(\mathbb{F}G)).$$

Now let $x=x_1+x_2b, y=y_1+y_2b \in \mathbb{F}G$ be arbitrary. Evidently, $y_1^{2^m}, x_2y_2^b + x_2^by_2 \in \zeta(\mathbb{F}G)$, $(y_1+y_1^b)^{2^m}=0$, and we obtain

$$[x,y,2^m+1]=[x_1+x_2b,y_1+y_2b,(y_1+y_2b)^{2^m}]=[(x_1+x_1^b)y_2b \\ +(y_1+y_1^b)x_2b+(x_2y_2^b+x_2^by_2)b^2,(y_1+y_1^b)^{2^m-1}y_2b]=0. \qquad \square$$

For brevity, we shall say that $x\in G$ is a δ-element provided for any $y_1,y_2\in G$, if $(x,y_1)\neq 1$, $(x,y_2)\neq 1$ and $\langle (x,y_1)\rangle \cap \langle (x,y_2)\rangle=\{1\}$ then $(y_1,y_2)\in\langle (x,y_1),(x,y_2)\rangle$. Clearly, any element with at most 2 conjugates is a δ-element.

Lemma 3. *Let G be nilpotent of class 2 such that its commutator subgroup G' is an elementary abelian 2-group of either finite order greater than 4 or of infinite order. Then the following statements are equivalent:*

(i) *there exists an abelian subgroup of index 2 in G;*
(ii) *any $x\in G$ is a δ-element;*
(iii) *there exists a δ-element $x\in G$ with $|G:C_G(x)|>2$.*

Proof. To prove (i)⇒(ii) let A be an abelian subgroup of index 2 in G. Since the centralizer of any element of A is of index at most 2 in G, any element of A is a δ-element. Pick $b\notin A$, $a_1b^k, a_2b^l\in G$ with $a_1,a_2\in A$, $k,l\in\{0,1\}$, and assume $(b,a_1b^k)=(b,a_1)\neq 1$, $(b,a_2b^l)=(b,a_2)\neq 1$, $(b,a_1)\neq(b,a_2)$. Then $(a_1b^k,a_2b^l)=(a_1,b^l)(b^k,a_2)\in\langle(b,a_1)\rangle\times\langle(b,a_2)\rangle$, and consequently b is a δ-element. Since there exists a conjugacy class in G of order greater than 2, (ii)⇒(iii) is even more obvious.

To prove the converse implications, first we make some observations. Assume that $x\in G$ with $|G:C_G(x)|>2$ is a δ-element, choose some subgroup D in (x,G) of order 4 and put $H_D=\{h\in G\mid (x,h)\in D\}$. We shall prove the following:

(I) $H'_D=D$;
(II) $(x,G)=G'$;
(III) $C_G(x)$ is abelian;
(IV) $(y,C_G(x))\subseteq\langle(y,x)\rangle$ for any $y\in G$, and all elements of $C_G(x)$ are δ-elements.

To prove (I) suppose on the contrary that there exist $v,w\in H_D$ such that $(v,w)\notin D$, and put $(x,v)=c$, $(x,w)=d$. Since x is a δ-element, either $c=1$, $d=1$ or $c=d$.

If $1\neq c=d$ then there exists $w_1\in H_D$ such that $(x,w_1)\notin\langle c\rangle=\langle d\rangle$. Obviously, we have $(v,w_1)\in D$ since x is a δ-element. But $(x,v)=c\in D$, $(x,ww_1)=c(x,w_1)\in D$ and $(v,ww_1)=(v,w)(v,w_1)\notin D$, a contradiction.

If one of the commutators c and d, say c, is 1, and the other, i.e. d, is not 1, then there exists $v_1 \in H_D$ such that $(x, v_1) \notin \langle d \rangle$. Since x is a δ-element, we see $(v_1, w) \in D$, but $(x, vv_1) = (x, v_1) \in D$, $(x, w) = d \in D$ and $(vv_1, w) \notin D$, a contradiction.

If $c = d = 1$ then there exist $v_2, w_2 \in H_D$ such that $(x, v_2) \neq 1$, $(x, w_2) \neq 1$ and $(x, v_2) \neq (x, w_2)$. By the previous case $(v_2, w_2), (v, w_2)$, $(v_2, w) \in D$, but $(x, vv_2) = (x, v_2) \in D$, $(x, ww_2) = (x, w_2) \in D$ and $(vv_2, ww_2) \notin D$, a contradiction proving (I).

To prove (II), if $v_3, w_3 \in G$ such that $(v_3, w_3) \notin (x, G)$ then choose some $E \subseteq (x, G)$ with $|E| = 4$ such that $(x, v_3), (x, w_3) \in E$. Putting $H_E = \{h \in G \mid (x, h) \in E\}$ we have $v_3, w_3 \in H_E$, contradicting (I).

Since $|G'| > 4$ and $(x, G) = G'$, there exist $E_1, E_2, E_3 \subseteq (x, G)$ with $|E_i| = 4$ such that $E_1 \cap E_2 \cap E_3 = \{1\}$. As $C_G(x) \subseteq H_{E_i}$ for $i = 1, 2, 3$, by (I) we easily infer (III).

While proving the first statement of (IV) we may suppose $y \notin C_G(x)$ because (IV) in this case gives just (III). Assume that there is $z \in C_G(x)$ such that $1 \neq (y, z) \notin \langle (y, x) \rangle$. Then there exists $c_1 \in G'$ with $c_1 \notin \langle (y, z) \rangle \times \langle (y, x) \rangle$, and putting $E = \langle c_1 \rangle \times \langle (y, x) \rangle$ we see that $y, z \in H_E$ contradicting (I). We proceed to show that all elements of $C_G(x)$ are δ-elements. Suppose on the contrary that there exist $z \in C_G(x)$ and $y_1, y_2 \in G$ such that $(z, y_1) \neq 1$, $(z, y_2) \neq 1$, $(z, y_1) \neq (z, y_2)$ and $(y_1, y_2) \notin \langle (z, y_1) \rangle \times \langle (z, y_2) \rangle$. The property (III) implies $y_1, y_2 \notin C_G(x)$, and by the first part of (IV) we deduce $(x, y_1) = (z, y_1)$, $(x, y_2) = (z, y_2)$, contradicting that x is a δ-element.

To show (iii)$\Rightarrow$(ii) pick some δ-element $u \in G$ with $|G : C_G(u)| > 2$ and some $g \in G$ which is not a δ-element. Then there exist $h_1, h_2 \in G$ such that $(g, h_1) \neq 1$, $(g, h_2) \neq 1$, $(g, h_1) \neq (g, h_2)$ and yet $(h_1, h_2) \notin \langle (g, h_1) \rangle \times \langle (g, h_2) \rangle$. Note that $h_1h_2, gh_1, gh_2, gh_1h_2$ are not δ-elements.

By (IV) we see $g, h_1, h_2 \notin C_G(u)$ i.e. $(u, h_1) \neq 1$, $(u, h_2) \neq 1$, $(u, g) \neq 1$. If $(u, h_1) = (u, h_2)$ then $h_1h_2 \in C_G(u)$, but h_1h_2 is not a δ-element and therefore $(u, h_1) \neq (u, h_2)$. Similarly, $(u, h_1) \neq (u, g)$, $(u, h_2) \neq (u, g)$ and $(u, h_1) \neq (u, g)(u, h_2)$. Since u is a δ-element, it follows $(g, h_1) \in \langle (u, g) \rangle \times \langle (u, h_1) \rangle$, $(g, h_2) \in \langle (u, g) \rangle \times \langle (u, h_2) \rangle$ and $(h_1, h_2) \in \langle (u, h_1) \rangle \times \langle (u, h_2) \rangle$.

We may choose h_1 and h_2 such that $(h_1, h_2) = (u, h_1)$. Indeed, this is clear in the case $(h_1, h_2) = (u, h_2)$, and if $(h_1, h_2) = (u, h_1)(u, h_2)$ then, putting $h_1' = h_1h_2$, we have $(u, h_1') = (h_1', h_2) = (u, h_1)(u, h_2)$.

Now (g, h_1) equals either (u, g) or $(u, g)(u, h_1)$. Since $(u, h_1) \neq (u, gh_2)$ and (h_1, gh_2) equals either (u, g) or $(u, g)(u, h_1)$, we see $(h_1, gh_2) \notin \langle (u, h_1) \rangle \times \langle (u, gh_2) \rangle$, contradicting that u is a δ-element.

There remained to prove (ii)$\Rightarrow$(i). Assume that (ii) holds, pick some $u \in G$ with $|G : C_G(u)| > 2$, choose some subgroup D in (u, G) of order 4 and put $H_D = \{h \in G \mid (u, h) \in D\}$.

We show that there exists an element in G with 2 conjugates. Suppose the contrary. Pick $b_1, b_2 \in H_D$ with $D = \langle (u, b_1)\rangle \times \langle (u, b_2)\rangle$, now b_1 and b_2 are δ-elements with more than 2 conjugates and $(b_1, b_2) \in D$. By (IV) we can see easily that this is impossible. Indeed, if $(b_1, b_2) = 1$ then $(u, b_2) \in \langle (u, b_1)\rangle$. If $(b_1, b_2) = (u, b_1)$ then $(b_1, ub_2) = 1$ and $(u, b_2) = (u, ub_2) \in \langle (u, b_1)\rangle$. If $(b_1, b_2) = (u, b_2)$ then $(b_2, ub_1) = 1$ and $(u, b_1) = (u, ub_1) \in \langle (u, b_2)\rangle$. Since ub_1 is not central in G, $|G : C_G(ub_1)| > 2$, and hence if $(b_1, b_2) = (u, b_1)(u, b_2)$ then $(ub_1, ub_2) = 1$ and $(u, b_2) = (u, ub_2) \in \langle (u, ub_1)\rangle$. In each of the four cases we arrived at a contradiction, thus there exists an element in G with 2 conjugates.

Pick some $a \in G$ with $|G : C_G(a)| = 2$. First we show that each element of $C_G(a) = A$ has at most only 2 conjugates in G. Let $G/A \cong \langle u_1 A\rangle$. If there exists $b \in A$ satisfying $|G : C_G(b)| > 2$ then, by (III), $C_G(b) \subseteq A$ because $a \in C_G(b)$. By (IV) for any $a_1 \in A$ we have $(u_1 a_1, a) \in \langle (u_1 a_1, b)\rangle$ and hence, first putting $a_1 = 1$, we obtain $(u_1, a) = (u_1, b)$ and $(u_1, a) = (u_1 a_1, b)$. Consequently, $(u_1^2 a_1, b) = (a_1, b) = 1$ and $C_G(b) = A$, a contradiction proving the desired property.

This readily follows $u \notin A$ and hence $G/A \cong \langle uA\rangle$. Finally, to show that A is abelian suppose on the contrary that there exist $a_2, a_3 \in A$ such that $(a_2, a_3) = c_3 \neq 1$. Then $A' = \langle c_3\rangle$ and $(u, a_2), (u, a_3) \in \langle c_3\rangle$. Since $(u, G) = (u, A) = G'$ by (II), there exist $a_4, a_5 \in A$ such that $(u, a_4) = c_4 \neq 1$, $(u, a_5) = c_5 \neq 1$, $c_4 \neq c_5$ and $c_3 \notin \langle c_4\rangle \times \langle c_5\rangle$. Observing $a_4, a_5 \in \zeta(A)$ we conclude that $(u, a_2 a_4)$ equals either c_4 or $c_3 c_4$, $(u, a_3 a_5)$ equals either c_5 or $c_3 c_5$, and $(a_2 a_4, a_3 a_5) = c_3 \notin \langle (u, a_2 a_4)\rangle \times \langle (u, a_3 a_5)\rangle$, contradicting that u is a δ-element. Thus A is an abelian subgroup of index 2 in G. □

Now we can complete the

Proof of Theorem 1. The result of Rips and Shalev mentioned in the introduction settles the case $p = 3$ and assures that if a noncommutative group algebra is 3-Engel then it is of characteristic 2 or 3. Hence to complete the proof consider the case $p = 2$.

Evidently, (ii) of Theorem 1 follows that $\mathbb{F}G$ is even Lie nilpotent of class at most 3. If (iii) of Theorem 1 holds then the central factor of G is of exponent 2 and Lemma 2 forces $\mathbb{F}G$ to be 3-Engel.

Now suppose that $\mathbb{F}G$ is a noncommutative 3-Engel group algebra of characteristic 2. By Lemma 1 $G/\zeta(G)$ is a group of exponent 2 and therefore elementary abelian, which follows that G is nilpotent of class 2 and its commutator subgroup is an elementary abelian 2-group. There remained to prove that if G' is either of finite order greater than 4, or of infinite order, then there is an abelian subgroup of index 2 in G. Suppose the contrary. Then by Lemma 3 there exists an element in G which is not a δ-element, i.e. there exists $g, h_1, h_2 \in G$ such that $(g, h_1) = c_1 \neq 1$, $(g, h_2) = c_2 \neq 1$, $c_1 \neq c_2$ and $(h_1, h_2) = c_3 \notin \langle c_1 \rangle \times \langle c_2 \rangle$. Since $\mathbb{F}G$ is 3-Engel, we have

$$\begin{aligned}[g, h_1 + h_2, 3] &= [g, h_1, h_1, h_1 + h_2] + [g, h_2, h_2, h_1 + h_2] + [g, h_1, h_2, h_2] \\ &\quad + [g, h_2, h_1, h_1] + [g, h_1, h_2, h_1] + [g, h_2, h_1, h_2] \\ &= [g, h_1, h_2, h_1] + [g, h_2, h_1, h_2] = gh_1^2 h_2 \widehat{c_1}\widehat{c_2 c_3}\widehat{c_3} + gh_1 h_2^2 \widehat{c_2}\widehat{c_1 c_3}\widehat{c_3} \\ &= gh_1 h_2 (h_1 + h_2)\widehat{c_1}\widehat{c_2}\widehat{c_3} = 0.\end{aligned}$$

It follows $h_1 \widehat{c_1}\widehat{c_2}\widehat{c_3} = h_2 \widehat{c_1}\widehat{c_2}\widehat{c_3}$, which is possible only if $h_1 \in h_2 \zeta(G)$, contradicting that h_1 and h_2 do not commute. □

Theorem 2. *Let $\mathbb{F}$ be a field of prime characteristic $p > 2$, G a non-abelian group. Then the group algebra $\mathbb{F}G$ is $(p+1)$-Engel if and only if G is nilpotent with a commutator subgroup of order p.*

Proof. The "if" claim is clear. To establish the "only if" claim assume that $\mathbb{F}G$ is $(p+1)$-Engel. Then, by Sehgal's theorem, G is nilpotent. Recall that for a normal subgroup N in G, $\mathbb{F}G/N \cong \mathbb{F}G/\mathcal{I}(N)$.

First we shall prove that for any $e, f \in G$ such that $(e, f) = c$ is of order p and central in $\langle e, f \rangle$ we have

$$(e + f)^p = e^p + f^p + \widehat{c} \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} e^k f^{p-k}.$$

Put $(e+f)^p = e^p + f^p + \sum_{k=1}^{p-1} w_k$, where w_k is the sum of all the products with e ocurring k times as a factor. Clearly, there are $\binom{p}{k}$ summands in w_k, and we can write $w_k = e^k f^{p-k} \sum_{i=1}^{\binom{p}{k}} c^{l_i}$. On the other hand, by Jacobson's formula [2, p.187] we have $w_k = s_k(e, f)$, where $ks_k(e, f)$ is the coefficient of λ^{k-1} in $[e, \lambda e + f, p-1]$, considered as a polynomial of the indeterminate λ, and hence, applying the identity $[x, y] = yx((x, y) - 1)$ several times, $w_k = \alpha e^k f^{p-k} \widehat{c}$ for some $\alpha \in \mathrm{GF}(p)$. By the first expression of w_k, the coefficient α cannot be else than $\frac{1}{p}\binom{p}{k}$.

Suppose that G is nilpotent of class 2 and $|G'| > p$. By Lemma 1 G' is of exponent p, and we may assume $|G'| = p^2$. It is easy to see that there exist $a \in G$ with more than p conjugates and $b_1, b_2 \in G$ such that $(a, b_1) = c_1 \neq 1$, $(a, b_2) = c_2 \notin \langle c_1 \rangle$ and $(b_1, b_2) = c_1$. Indeed, this is clear if (b_1, b_2) is in $\langle c_1 \rangle$ or $\langle c_2 \rangle$. If $(b_1, b_2) = c_1^k c_2^l$ with $1 \leq k, l \leq p-1$ then, putting $b_1' = b_1^k b_2^l$, $b_2' = b_2^{k'}$, where $kk' \equiv 1 (\mathrm{mod}\, p)$, we see $(a, b_1') = c_1^k c_2^l \neq 1$, $(a, b_2') = c_2^{k'} \notin \langle (a, b_1') \rangle$ and $(b_1', b_2') = (b_1, b_2)^{kk'} = (a, b_1')$.

If $p = 3$ then

$$\begin{aligned}
[a, b_1 + b_2, 4] &= [a, b_1 + b_2, (b_1 + b_2)^3] \\
&= [a, b_1 + b_2, b_1^3 + b_2^3 + \widehat{c_1}(b_1 b_2^2 + b_1^2 b_2)] \\
&= \widehat{c_1}[a, b_2, b_1 b_2^2 + b_1^2 b_2] = \widehat{c_1}\big(b_1[a, b_2, b_2^2] + b_1^2[a, b_2, b_2]\big) \\
&= \widehat{c_1} b_1 (b_1 - b_2)[a, b_2, b_2] = \widehat{c_1}\widehat{c_2}(b_1 - b_2) a b_1 b_2^2 = 0,
\end{aligned}$$

which implies $b_1 \in b_2 \zeta(G)$, a contradiction. If $p > 3$ then put $z = (c_2 - 1)^{\frac{p-1}{2}}$ and compute

$$\begin{aligned}
[a, zb_1 + b_2, p + 1] &= [a, zb_1 + b_2, (zb_1 + b_2)^p] \\
&= [a, zb_1 + b_2, b_2^p + \widehat{c_1}(zb_1 b_2^{p-1} + \frac{p-1}{2} z^2 b_1^2 b_2^{p-2})] \\
&= \widehat{c_1}[a, b_2, zb_1 b_2^{p-1} + \frac{p-1}{2} z^2 b_1^2 b_2^{p-2}] \\
&= \widehat{c_1} z b_1 [a, b_2, b_2^{p-1}] = 0.
\end{aligned}$$

Since $0 = [x, b_2^p] = [x, b_2] b_2^{p-1} + b_2 [x, b_2^{p-1}]$, it follows

$$\widehat{c_1} z [a, b_2, b_2] = \widehat{c_1}(c_2 - 1)^{\frac{p+3}{2}} b_2^2 a = 0,$$

which is impossible since $\frac{p+3}{2} < p$. Thus if G is nilpotent of class 2 then $|G'| = p$.

Now suppose that G is nilpotent of class greater than 2. Since $G/\gamma_4(G)$ is of class 3, we may assume that G is of class 3. Since $G/\gamma_3(G)$ is of class 2, the facts proved above follow that $G'/\gamma_3(G)$ is of order p. By Lemma 1 $\gamma_3(G)$ is of exponent p and, since in a group nilpotent of class 3 we have $(x, y^p) = (x, y)^p (x, y, y)^{p\frac{p-1}{2}}$ and p is odd, G' is of exponent p. Combining these observations we may suppose that G is of class 3 with an elementary abelian commutator subgroup of order p^2.

If G is not a 2-Engel group then there exist $g, h \in G$ such that $(g, h) = d \notin \gamma_3(G)$, $1 \neq (d, h) = c \in \gamma_3(G)$ and $(d, g) = 1$. Indeed, if $(g, h, g) = c^r$

with $1 \le r \le p-1$ then $(h^{-r}g, h) = (g, h) = d$, $(d, h^{-r}g) = c^{-r}c^r = 1$. We have

$$[g, d+h, p+1] = [g, d+h, (d+h)^p]$$
$$= \left[g, d+h, 1+h^p + \widehat{c}\sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k}d^k h^{p-k}\right] = \widehat{c}\sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k}d^k[g, h, h^{p-k}]$$
$$= \widehat{c}\sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k}d^k h^{p-k+1}g(d^{p-k}-1)(d-1) = 0.$$

Multiplying by $(d-1)^{p-3}$ it follows

$$\widehat{c}\widehat{d}gh\sum_{k=1}^{p-1}\frac{1}{p}\binom{p}{k}(p-k)h^{p-k} = \widehat{c}\widehat{d}gh\sum_{k=1}^{p-1}\binom{p-1}{k}h^{p-k}$$
$$= \widehat{c}\widehat{d}gh\sum_{k=1}^{p-1}(-1)^k h^{p-k} = \widehat{c}\widehat{d}gh^2\sum_{k=0}^{p-2}(-1)^k h^k = 0,$$

which is possible only if $h \in \langle c, d\rangle$, contradicting $(d, h) \ne 1$.

Now we prove the following auxiliary assertion: if H is a 2-Engel group nilpotent of class 3 then $|H'/\gamma_3(H)| \ne 3$. Indeed, suppose the contrary. Then there exist $u, v \in H$ such that $H'/\gamma_3(H) = \langle (u,v)\gamma_3(H)\rangle$, and, furthermore, there exists $w \in H$ with $(u, v, w) \ne 1$. Clearly, either (u, w) or (v, w), say (u, w), is noncentral in H, and hence $(u, w) = (u, v)^k z$, where $k \in \{1, 2\}$ and z is central in H. However, since in the 2-Engel group H the exponent of $\gamma_3(H)$ is 3, it follows $1 = (u, w, w) = ((u, v)^k z, w) = (u, v, w)^k \ne 1$, a contradiction.

Finally, in the case when G is 2-Engel and of class 3 we have $p = 3$ and $|G'/\gamma_3(G)| = 3$, which is impossible by the previous auxiliary assertion. $\square$

Theorem 3. *If $\mathbb{F}G$ is a 3-Engel group algebra then the group of units $U(\mathbb{F}G)$ is 3-Engel.*

Proof. The implication is evident if (i), (ii) or (iv) of Theorem 1 holds. Suppose that $p = 2$ and G is nilpotent of class 2 such that its commutator subgroup is an elementary abelian 2-group of order either finite greater than 4, or of infinite order, and there exists an abelian subgroup A of index 2 in G. We shall use the notations and observations made in the proof of Lemma 2. For arbitrary noncommuting units

$x = x_1 + x_2 b, y = y_1 + y_2 b \in U(\mathbb{F}G)$ we shall prove $(x, y, 3) = 1$ by means of the identity $(x, y, y^2) = (x, y, y)^2 (x, y, 3)$. Obviously,

$$t = x_1 + x_1^b,\ w = y_1 + y_1^b,\ z = x_2 y_2^b + x_2^b y_2 \in \zeta(\mathbb{F}G),\quad t^2 = w^2 = z^2 = 0,$$

$I = t\mathbb{F}G + w\mathbb{F}G + z\mathbb{F}G$ is an ideal, $I^2 = tw\mathbb{F}G + tz\mathbb{F}G + wz\mathbb{F}G$, $I^4 = \{0\}$, and

$$[x, y] = ty_2 b + wx_2 b + zb^2 \in I,\quad y^2 \equiv wy_2 b (\operatorname{mod} \zeta(\mathbb{F}G)),$$
$$[x, y, y] = [x, y^2] = w(ty_2 b + zb^2) \in I^2.$$

Moreover,

$$(x, y) = 1 + x^{-1} y^{-1} [x, y] \in 1 + I,\quad (x, y)^2 \in 1 + I^2,$$
$$(x, y^2) = 1 + x^{-1} y^{-2} [x, y^2] \in 1 + I^2,$$

which, since $(x, y^2) = (x, y)^2 (x, y, y)$, immediately follows $(x, y, y) \in 1 + I^2$, and hence $(x, y, y)^2 = 1$, $(x, y, 3) = (x, y, y^2)$. Recalling $[x, y, 3] = [x, y, y^2] = 0$ we conclude

$$[(x, y), y^2] = [x^{-1} y^{-1} [x, y], y^2] = x^{-1} y^{-1} [x, y, y^2] + [x^{-1}, y^2] y^{-1} [x, y]$$
$$= y^{-1} x^{-1} [x, y^2] x^{-1} [x, y] = y^{-1} x^{-1} w(ty_2 b + zb^2) x^{-1} (ty_2 b + wx_2 b + zb^2)$$
$$= y^{-1} x^{-1} twz [x^{-1}, y_2 b] b^2 = y^{-1} x^{-1} x^{-1} twz [x, y_2 b] x^{-1} b^2 = 0. \qquad \square$$

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