Structure theorems for objects

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§ 1. Introduction

From a categorical point of view the notion of structure theorems can be defined as follows. Let us distinguish a class $\mathcal{M}$ of objects. These objects will be regarded as simple objects. Let us choose a decomposition procedure $\mathcal{D}$ (e.g. decomposition in product, coproduct, subdirect embeddings, direct or inverse limit, etc.). A theorem is a structure theorem, if it asserts that a class of objects can be decomposed into objects of $\mathcal{M}$ by the procedure $\mathcal{D}$. Because of the ring-theoretical analogy, a structure theorem may be called of *Wedderburn—Artinian type* if either 1) the objects of $\mathcal{M}$ are simple in the usual algebraic sense and $\mathcal{D}$ means decomposition in product, or 2) the theorem establishes a categorically dual representation to that of 1).

It is the purpose of this paper to develop a structure theorem of Wedderburn—Artinian type in a category satisfying some rather natural additional requirements, and to give its interpretations in several concrete categories. After the preliminary section § 2, in § 3 subdirect embeddings of semisimple objects will be established (Theorem 1). It is remarkable that in the proof of the subdirect embedding theorem, we do not use any strong, particularly algebraic conditions (in the proof of BIRKHOFF's famous subdirect embedding theorem, it is essentially used that the lattice of congruence-relations of an algebra is an algebraic one [2], [21].).

Using two conditions (one of them is closely related to GROTHEDIECK's [8] axiom AB 5) we obtain our structure theorem of Wedderburn—Artinian type (Theorem 2) which establishes a decomposition of certain semisimple objects in a product of simple objects. However, it would be possible to introduce a kind of radical (in the sense of [18] and [20]), semisimplicity properties, as well as radical properties will not be investigated here.

Several applications of Theorems 1 and 2 will be given in § 4. Hereby we get a common categorical aspect of facts belonging to algebra, algebraic logic, functional analysis and general topology. It is perhaps in some extent interesting that from this point of view the ring-theoretical and semigroup-theoretical structure theorems of Wedderburn—Artinian type are duals of each other.

§ 2. Preliminaries

In general our terminology is based on MITCHELL's book [15].

An object S of a category will be called a *singleton*, if (a) for every object A there exists at least one morphism $\alpha: S \to A$, and (b) for every object B there exists exactly one morphism $\beta: B \to S$. All singletons of a category are equivalent. Cosingletons are defined dually (cf. SEMADENI [16]).

Conormal quotient objects will be called *factorobjects*.

Throughout § 2 and § 3 it will be supposed that the considered category $\mathscr{C}$ satisfies the following axioms:

(A_1) *Either $\mathscr{C}$ has a singleton S and the class* Map (S, A) *of all morphisms $S \to A$ is a set, for all $A \in \mathscr{C}$, or $\mathscr{C}$ has a cosingleton S^* and* Map (A, S^*) *is a set for all $A \in \mathscr{C}$;*

(A_2) *Every morphism $\alpha \in \mathscr{C}$ factors as $\alpha = \nu\mu$ by an epimorphism μ and a monomorphism ν* (the existence of images is not supposed).

(A_3) *$\mathscr{C}$ is colocally small* in the sense that every $A \in \mathscr{C}$ has a representative class of factorobjects which is a set;

(A_4) *For every family $\{A_i\}_{i \in I}$ of factorobjects of an object A the counion $\bigcup\limits_{i \in I}^* A_i$ exists*[1]) *and it is again a factorobject of A;*

(A_5) *For any two factorobjects A, B of an object C the cointersection $A \cap^* B$ exists and it is again a factorobject.*

(A_6) *For every family $\{A_i\}_{i \in I}$ of objects, the product $\prod\limits_{i \in I} A_i$ exists and the projections $\pi_i : \prod\limits_{i \in I} A_i \to A_i$, are conormal epimorphisms.*

Suppose that S is a singleton, and denote by ν_A the only morphism $A \to S$. According to (A_1), by the axiom of choice we can select exactly one morphism ω_B of each set Map (S, B). Let us define $\omega_{AB} = \omega_B \nu_A : A \to B$. For any objects A, B, C we have $\omega_{AC} = \omega_{BC}\omega_{AB}$. Similar consideration can be made in the case if S is a cosingleton (cf. [16]).

In (A_6) the second requirement, namely the projections should be conormal epimorphisms, is fulfilled *if the projections have conormal images*, i.e. π_i can be factored as $\pi_i = \nu\mu$ where μ is a conormal epimorphism and ν is a monomorphism (cf. 13] § 14. 2).

A variant of the dual statement of GROTHENDIECK's [8] axiom AB 5 is condition

(C^*). *Let $\{A_i\}_{i \in I}$ be an inverse system of factorobjects of an object A. Then $\bigcup^* A_i = \varprojlim \{A_i\}_{i \in I}$ holds.*

We shall make use of condition

(D^*) *If A and B are two factorobjects of an object C such that $A \cup^* B = C$ and $A \cap^* B = S$ then the canonical morphism $\gamma: C \to A \times B$ is an epimorphism.*

Let us mention that by BIRKHOFF [2] VI. Theorem 4 the categories of groups, rings, etc. fulfil condition (D^*). Moreover, the category $\mathscr{C}_s^*$ of semigroups with zero satisfies the dual condition (D) of (D^*). (In $\mathscr{C}_s^*$ the coproduct means the disjoint union with identified 0 elements).

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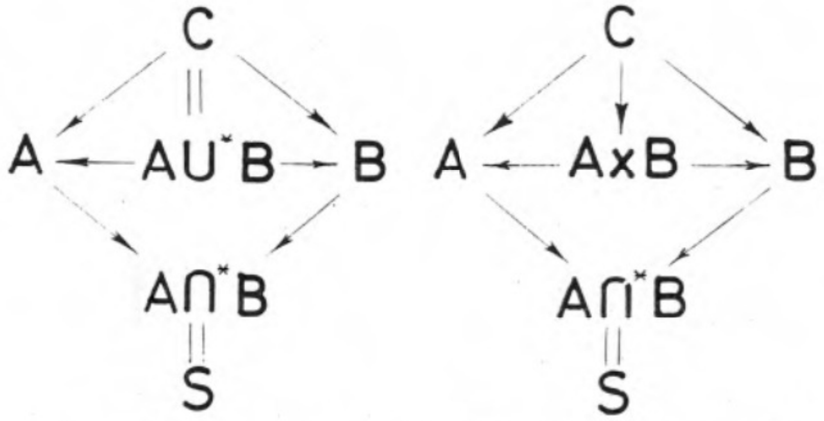
[1]) The counion $\cup^* A_i$ means a minimal quotient object which contains every A_i as a quotient object. By (A_4) it is uniquely determined up to isomorphism.

Proposition. *If the category $\mathscr{C}$ satisfies condition (D^*) then the counion $A \cup {}^* B$ of the factorobjects A, B is isomorphic to the product $A \times B$.*

Consider, namely, the diagrams



By the assumptions there exist unique epimorphisms $\varphi: A \cup^* B \to A \times B$ and $\psi: A \times \times B \to A \cup^* B$ such that commutativity is preserved in the diagrams. Hence $\psi\varphi$ and $\varphi\psi$ have to be the identities of $A \cup^* B$ and $A \times B$, respectively, and the isomorphism is proved.

An object A is said to be *subdirectly embedded* in the product $\prod_{i \in I} A_i$, if there is a monomorphism $\gamma: A \to \prod_{i \in I} A_i$ such that all morphisms $\pi_i \gamma: A \to A_i$ are conormal epimorphisms where π_i denotes the projection $\prod_{i \in I} A_i \to A_i$ (cf. [18]).

§ 3. Structure theorem for semisimple objects

Let $\mathscr{M}$ be a class of objects of the category $\mathscr{C}$ such that $\mathscr{M}$ satisfies

(i) *if $A \in \mathscr{M}$ and $A \approx B$, then $B \in \mathscr{M}$ follows;*

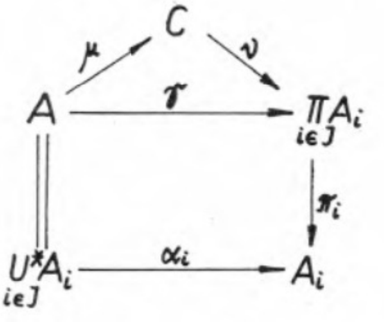
(ii) *if $A \in \mathscr{M}$ and $\alpha: A \to B$ is a conormal epimorphism, then either $A \approx B$ or $\alpha = \omega_{AB}$.*

Condition (i) means that $\mathscr{M}$ defines an abstract property of objects, and (ii) can be regarded as the definition of simplicity of objects.

For any object A let us consider the set $M_A = \{A_i | i \in I\}$ of all (non isomorphic) factorobjects of A, belonging to $\mathscr{M}$. We shall say that M_A is the *structure $\mathscr{M}$-space of A*, and the factorobjects belonging to $\mathscr{M}$, will be called *$\mathscr{M}$-factorobjects*. For any object A the counion $\bigcup^*_{i \in I} A_i$ of $\mathscr{M}$-factorobjects, will be called the *$\mathscr{M}$-semisimple image of A*. If M_A is empty, then by the $\mathscr{M}$-semisimple image one understands a singleton (or cosingleton). The $\mathscr{M}$-semisimple image of an object A will be denoted by $\mathscr{S}_M(A)$. An object A will be said to be *$\mathscr{M}$-semisimple*, if $\mathscr{S}_M(A) = A$ holds.

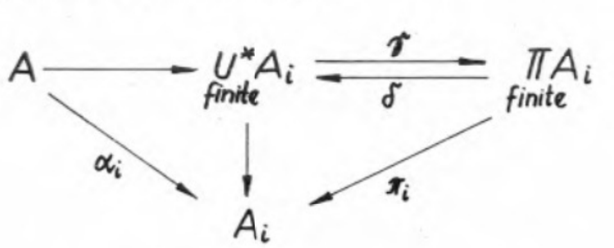
Theorem 1. *Every $\mathscr{M}$-semisimple object can be subdirectly embedded in a product of objects belonging to $\mathscr{M}$. More precisely, if $M_A = \{A_i | i \in I\}$ is the structure $\mathscr{M}$-space of A, then the canonical morphism $\gamma: A \to \prod_{i \in I} A_i$ is a monomorphism.*

PROOF. It is sufficient to show that γ is a monomorphism. According to (A_2) γ can be factored as $\gamma=\nu\mu$ by an epimorphism μ and monomorphism ν such that the diagram



is commutative for all $i\in I$. Hence $\pi_i\nu\mu=\alpha_i$ implies that $\pi_i\nu$ is a conormal epimorphism for all $i\in I$. Thus there exists a unique epimorphism $\varrho: C\to\bigcup^*_{i\in I} A_i = A$, moreover, from $\varrho\mu=\varepsilon_A$ it follows that μ is a monomorphism. Thus also γ is a monomorphism.

An *independent $\mathcal{M}$-system* $\Delta=\{A_i|i\in I\}$ of A is a system consisting of $\mathcal{M}$-factorobjects of A such that the canonical morphism $\gamma\colon \bigcup^*_{\text{finite}} A_i \to \prod_{\text{finite}} A_i$ is an epimorphism for every finite subset of I. First of all, let us observe that γ is, of course, an isomorphism, since for $\alpha: A\to \bigcup^*_{\text{finite}} A_i$ we have that $\gamma\alpha$ is an epimorphism and so by the definition of counion, there exists an epimorphism $\delta\colon \prod_{\text{finite}} A_i \to \bigcup^*_{\text{finite}} A_i$ with commutativity preserved in the diagram



Hence $\pi_i\gamma\delta=\pi_i$ is valid for all i of the finite subset, and this implies that $\gamma\delta$ is the identity morphism of $\prod_{\text{finite}} A_i$. Thus δ is a monomorphism. On the other hand δ is an epimorphism, and it has to be conormal, since $\alpha=\delta(\gamma\alpha)$ is conormal and $\gamma\alpha$ is an epimorphism. Thus δ and so also γ are isomorphisms.

Let $\Delta_1\subset\cdots\subset\Delta_\lambda\subset\cdots$ be an ascending chain of independent $\mathcal{M}$-systems of an object A, and denote the join of them by Δ. Obviously, Δ is again an independent $\mathcal{M}$-system. Hence by (A_3), Zorn's lemma guarantees the existence of a maximal independent $\mathcal{M}$-system of any object A.

Lemma. *Assume that the category $\mathcal{C}$ fulfils also condition* (D^*). *Let $\Delta_1=\{A_i|i\in I\}$ and $\Delta_2=\{B_j\,|\,j\in J\}$ be independent $\mathcal{M}$-systems of an object A. Then $\bigcup^*_{i\in I} A_i = \bigcup^*_{i\in J} B_j$ is valid* (i.e. *they are isomorphic factorobjects*).

PROOF. Assume that the statement is not true. Now, without loss of generality, we can suppose $\bigcup^*_{i\in I} A_i \not\equiv \bigcup^*_{i\in J} B_j$. So there exists at least one $B_j \in \Delta_2$ such that $\bigcup^*_{i\in I} A_i \not\equiv B_j$. For any finite subset $N=\{i_1, \dots, i_n\}$, clearly, $\bigcup^*_{i\in N} A_i \not\equiv B_j$ holds. Since Δ_1 is an independent $\mathcal{M}$-system, therefore it follows $\bigcup^*_{i\in N} A_i = \prod_{i\in N} A_i$. Since $B_j \in \mathcal{M}$, so by (ii) $B_j \cap^* (\prod_{i\in N} A_i)$ has to be a singleton (or cosingleton). Applying the Proposition, we obtain that $B_j \cup^* (\prod_{i\in N} A_i) = B_j \times (\prod_{i\in N} A_i)$. Thus $\Delta = \{B_j, A_i | i\in I\}$ forms an independent $\mathcal{M}$-system which contradicts the maximality of Δ_1.

As an immediate consequence we get a

COROLLARY. *Let $\Delta = \{A\ | i\in I\}$ be a maximal independent $\mathcal{M}$-system of an object A, then $\bigcup^*_{i\in J} A_i = \mathcal{S}(A)$*

Assume that the considered category satisfies also condition (D^*). To an arbitrary object A we can define an inverse system Ω_A as follows. Consider a maximal independent $\mathcal{M}$-system $\Delta = \{A_\lambda | \lambda \in \Lambda\}$, and let $F(\Lambda)$ denote the set of all finite subset of Λ. The products $\prod_{i\in I} A_i$, $I \in F(\Lambda)$ form an inverse system Ω_Δ with the inverse mapping system: for all $I, K \in F(\Lambda)$ and $K \subseteqq I$, the morphism π^I_K should be the canonical morphism $\pi^I_K\colon \prod_{i\in I} A_i \to \prod_{k\in K} A_k$. From the definition of product and inverse limit it follows immediately $\prod_{\lambda\in\Lambda} A_\lambda = \varprojlim \Omega_\Delta$. Let us remark that by the Proposition and (ii) $\bigcup^*_{i\in I} = \prod_{i\in I} A_i$, $I \in F(\Lambda)$, holds, further according to the Lemma and its Corollary, Ω_Δ is a cofinal subsystem of $\Omega_A = \{\bigcup^*_{\text{finite}} A_r | A_i \in M_A\}$ which forms an inverse system itself. Thus $\varprojlim \Omega_A = \varprojlim \Omega_\Delta$ holds, and it does not depend (up to isomorphism) from the choice of Δ.

An object A will be called *$\mathcal{M}$-compact*, if the canonical morphism $\gamma\colon A \to \varprojlim \Omega_A$ is an epimorphism. For an $\mathcal{M}$-compact object A obviously $\varprojlim \Omega_A = \bigcup^*_{A_i\in M_A} A_i$ holds, and so $\mathcal{M}$-compactness is a special kind of condition (C^*). As it was mentioned in § 2, condition (C^*) is a variant of the dual condition of GROTHENDIECK's Axiom AB 5. It is well known that every variety of algebras does satisfy AB 5, but it does not fulfil the dual condition of AB 5. In the algebra $\mathcal{M}$-compactness has a topological meaning, and it is essentially a generalization of LEPTIN's "linear compactness in the narrow sense" ([14], [20]; see also § 4).

Theorem 2. *Assume the category satisfies condition* (D^*). *The object A is $\mathcal{M}$-compact and $\mathcal{M}$-semisimple if and only if $A = \prod_{\lambda\in\Lambda} A_\lambda$ where $\Delta = \{A_\lambda | \lambda\in\Lambda\}$ forms a maximal independent $\mathcal{M}$-system of A.*

PROOF. By the Corollary we have

$$A = \mathcal{S}_M(A) = \bigcup^*_{\lambda\in\Lambda} A_\lambda = \varprojlim \{\bigcup^*_{\text{finite}} A_\lambda\} = \prod_{\lambda\in\Lambda} A_\lambda .$$

Restricting the considerations to all cardinals smaller than a given one in (A_3) and (A_5) as well as for the cardinality of M_A, we obtain obvious variants of Theorems 1 and 2.

§ 4. Some special cases

In this section we present several applications of Theorems 1 and 2, as well as their duals. The special categories given below, are chosen from the algebra, functional analysis, algebraic logic and general topology. The given examples, of course, will not exhaust all the possibilities of the applications. Moreover, we will concern only interpretations of our results, further investigations in special concrete categories, will not accomplish.

1. Specialised category-theoretical investigations with strong ring- and module-theoretical background were made by SULIŃSKI [18], KWANGIL KOH [12] and the author [20]. Theorem 1 infers SULIŃSKI's [18] Theorem 4. 9 and Theorem 2 yields Theorem 5. 6 of [20].

2. Let $\mathscr{C}_R$ be the *category of rings*. Denote the class of all simple rings with unity by $\mathscr{M}$. The $\mathscr{M}$-semisimple rings are exactly those *having zero Brown—McCoy radical*. If a ring is semisimple and it satisfies the descending chain condition, then Theorem 2 yields the *classical Wedderburn—Artin Structure Theorem*.

3. Let $\mathscr{C}_M$ denote the *category of (left) modules over a ring R*. An R-module M is irreducible, if it is simple and $RM\neq 0$. Consider the class $\mathscr{M}$ of all irreducible R-modules. Now an R-module A is $\mathscr{M}$-semisimple precisely if the intersections of all its submodules L with $A/L\in\mathscr{M}$, is zero; i.e. *A has zero Kertész radical* ([10], and [11] p. 141). The ring R is semisimple in the sense of KERTÉSZ, if it is a semisimple R-module. The Kertész radical of a ring need not coincide with the Jacobson radical (cf. SZÁSZ [19]), but if R has right unity, then they coincide. Let R be a ring with right unity. The finite intersections of left ideals L_i with $R/L_i\in\mathscr{M}$, $i\in I$, form a filter which induces a linear topology in R. This topology is Hausdorffian if and only if R is semisimple. If R is linearly compact (see LEPTIN [14]) then this topology is complete and so by ZELINSKY [22] the canonical map $\gamma: R/\bigcap_{i\in I} L_i \to \varprojlim \{R/\bigcap_{\text{finite}} L_j\}$ is an isomorphism and a homeomorphism too. Thus *the linearly compact Jacobson semisimple ring R is $\mathscr{M}$-compact and $\mathscr{M}$-semisimple*. Hence Theorem 2 infers the *Leptin—Noether-decomposition* of semisimple linearly compact rings with right unity in a complete direct sum of irreducible R-modules [14].

4. Let $\mathscr{C}_N$ denote the *category of N-groups over a near ring N*. Decomposition theorems and a radical were developed by BLACKETT [2] and BETSCH [1]. As in § 3, Theorem 1 and 2 yield again corresponding statements.

5. Consider *the category $\mathscr{C}_B$ of commutative Banach algebras*, and let $\mathscr{M}$ consist of the field of complex numbers. $\mathscr{M}$-semisimplicity means just the usual one, and so by Theorem 1 *the elements of a semisimple commutative Banach algebra are complex valued functions*. (C^*-algebras are semisimple; for details we hint e.g. to [7].)

6. Let $\mathscr{C}_D$ denote *the category of distributive lattices*, and let $\mathscr{M}$ consist of the two-element lattice (0, 1). For any elements d_1, d_2 of a distributive lattice D there exists a conormal epimorphism $\varphi: D\to(0, 1)$ such that $\varphi(d_1)=0$ and $\varphi(d_2)=1$. This implies that every distributive lattice is $\mathscr{M}$-semisimple. Thus Theorem 1 yields that *every distributive lattice (and so also every Boolean algebra) can be embedded in the lattice of all subsets of a set* (cf. BIRKHOFF [2]).

7. Consider the *category $\mathscr{C}_P$ of polyadic Boolean algebras*. Let $\mathscr{M}$ consist of simple polyadic algebras. Now $\mathscr{M}$-semisimplicity means the usual one. Since every

polyadic algebra is semisimple, so Theorem 1 infers *the algebraic version of* GÖDEL's *completeness theorem*. (For details we refer to HALMOS [9].)

8. Let $\mathscr{C}_T$ be *the category of topological spaces*. Put $\mathscr{M}=\{(0, 1)\}$ where $(0, 1)$ denotes the discrete two-point space. If $T\in\mathscr{C}_T$ is a 0-dimensional space, then T has a base $U=\{U_i\}_{i\in I}$ consisting of open- and closed-sets and the mappings

$$\varphi_i(x)=\begin{cases}1 & \text{if} \quad x\in U_i\\ 0 & \text{if} \quad x\notin U_i\end{cases}$$

separate the points of T. Hence the 0-dimensional spaces are $\mathscr{M}$-semisimple. Thus Theorem 1 implies that the *Cantor cube* $D^{\mathfrak{m}}=\prod_{i\in I}(0, 1)$ is *a universal space for 0-dimensional spaces of weight* $|I|=\mathfrak{m}\geqq\aleph$ (cf. ENGELKING [5]).

9. Let $\mathscr{C}_s^*$ denote the *category of semigroups* with zero, and let $\mathscr{M}^*$ be the class of completely 0-simple semigroups (a semigroup S with zero is completely 0-simple, if it has only trivial ideals, $S^2\neq 0$, moreover S contains 0-minimal left and right ideals, cf. CLIFFORD—PRESTON [4]). Now $\mathscr{C}_s^*$ fulfils all the dual conditions of axioms (A_1)—(A_6) as well as $\mathscr{M}^*$ satisfies the duals of (i) and (ii). Since $\mathscr{C}_s^*$ satisfies AB 5, so every semigroup is dually $\mathscr{M}^*$-compact, and also the dual condition (C) of (C^*) is fulfilled, further also (D) holds. (Coproduct means the disjoint union with identified 0 element.) Thus the dual assertion of Theorem 2 establishes a *decomposition in disjoint union of completely 0-simple semigroups for all co-$\mathscr{M}^*$-semisimple semigroups*. (Further classifications of co-$\mathscr{M}^*$-semisimple semigroups have been given by STEINFELD [17].) However, this statement is trivial it seems to be essentiall for some further developments that from this point of view the decomposition theory of semigroups is dual to that of rings. Up to this time there were made many attempts to obtain results for semigroups which are analogous to ring-theoretical results concerning the multiplicative semigroup of rings (e.g. there were introduced several kinds of radicals for semigroups). This duality gives a new aspect of the relation between ring and semigroup theory.

10. Let $\mathscr{C}_c^*$ denote the *category of compact Hausdorff spaces*. The one point space is a singleton in $\mathscr{C}_c^*$, further all the dual conditions of (A_2)—(A_6) are satisfied (in (A_4) and (A_6) the Čech—Stone compactification of the union and disjoint union should be considered, respectively). Let $\mathscr{M}^*$ consist of the one point space. Since every space of $\mathscr{C}_c^*$ is co-$\mathscr{M}^*$-semisimple, so the dual assertion of Theorem 1 yields the well known fact that *every compact Hausdorff space is a continuous image of the Čech—Stone compactification of a discrete space* (cf. e.g. [16]).

11. *Let $\mathscr{C}_A^*$ be the category of abelian groups*, and let $\mathscr{M}^*$ consist of a cyclic p-group for a fixed prime p. The co-$\mathscr{M}^*$-semisimple groups are just the elementary p-groups, and by (C) and (D) the dual assertion of Theorem 2 yields *the decomposition of elementary p-groups in a discrete direct sum of cyclic p-groups* (cf. FUCHS [6]).

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