On the asymptotic behaviour of the solutions of $(p(t)x')+q(t)f(x)=0$

By L. HATVANI (Szeged)

1. Introduction

In this paper we give conditions that either *every oscillatory* solution or *every non-oscillatory* solution $x(t)$ of the differential equation

$$(p(t)x')'+q(t)f(x)=0 \tag{E}$$

satisfies the relation

$$\lim_{t\to\infty} x(t)=0. \tag{R_1}$$

Applying results concerning oscillatory properties of (E) ([8], [9]), by the aid of our theorems conditions can be given which assure that *every* solution of (E) satisfies the relation (R_1). D. WILLETT and J. S. W. WONG have presented such a condition in [1] (Theorem 1. 1). Our results are independent of the theorem mentioned above regarding of both the method and the applicability of the theorems, as it will be shown by the examples.

In our paper we present conditions even for *global asymptotical stability (g. a. s.) of the zero solution* of (E), which means that every solution $x(t)$ of (E) satisfies the relation

$$\lim_{t\to\infty} x(t)=\lim_{t\to\infty} x'(t)=0. \tag{S}$$

In Sec. 1 we study the continuation and boundedness of the solutions. In Sec. 2 we establish a *necessary* condition that every solution $x(t)$ satisfies (R_1), and prove that this condition is also *sufficient* for the same property *for every non-oscillatory* solution. In Sec. 3 we investigate the *oscillatory* solutions. In Sec. 4 we give a *necessary* condition for the g. a. s. of the zero solution of (E) and a *sufficient* condition for the same property. In Sec. 5 we apply our results to the equation

$$x''+a(t)x'+b(t)f(x)=0, \tag{E'}$$

moreover we study the connections of our theorems with one another and with some known theorems respectively.

1. Notations and necessary lemmas

Suppose in the following that

(A_1) $p(t)\in C'[T,\infty)$, $q(t)\in C'[T,\infty)$; $p(t)>0$, $q(t)>0$ on an appropriate interval $[T,\infty)$;

(A_2) $f(x)\in C(-\infty,\infty)$; $xf(x)>0$ for all $x\neq 0$, and $\lim\limits_{|u|\to\infty} F(u)=\infty$, where $F(u) = \int_0^u f(x)\,dx$;

(A_3) for arbitrary $t_0(\geqq T)$, x_0, x_0' (E) has a unique solution $x(t)=x(t;t_0,x_0,x_0')$ in an appropriate interval $(t_0-\xi,\ t_0+\xi)$, $(\xi>0)$ with $x(t_0)=x_0$ and $x'(t_0)=x_0'$.

Let $x(t)$ be an arbitrary solution of (E) and introduce the following notations:

$$V(t,u,v)=\frac{p(t)}{q(t)}v^2+2F(u),$$

(1.1)

$$v(t)=V(t,x(t),x'(t))=\frac{p(t)}{q(t)}[x'(t)]^2+2F(x(t)).$$

Lemma 1.1. *If*

$$\int_T^\infty \left[\left(\ln\left(p(t)q(t)\right)\right)'\right]_-\,dt<\infty, \tag{1.2}$$

then

a) *every solution $x(t;t_0,x_0,x_0')$ of* (E) *exists in $[T,\infty)$;*

b) *$v(t)$ is a function of bounded variation on $[T,\infty)$, and consequently it tends to a finite limit as $t\to\infty$.*

c) *$x(t)$ and $[p(t)]^{\frac12}[q(t)]^{-\frac12}x'(t)$ are bounded.*

Remark 1.1. Carry out the transformation

$$u=\int_T^t \frac{1}{p(s)}\,ds, \tag{1.3}$$

then the equation (E) becomes

$$\frac{d^2x}{du^2}+p(t)q(t)f(x)=0, \tag{1.4}$$

which emphasizes the importance of the function $p(t)q(t)$ in the lemma and through this paper. But the transformation (1.3) can be used only if $\int_T^\infty [p(s)]^{-1}ds=\infty$, which is not supposed in our theorems, thus our results can't be deduced in general from known theorems concerning the equation of the type (1.4).

PROOF. Suppose that $x(t;t_0,x_0,x_0')$ is a solution of (E) and $[t_0,T_1)$ is the maximum interval to the right in which the solution $x(t)$ can be continued, $(T\leqq t_0<T_1\leqq\infty)$. Using (E) it is easy to see that

$$v'(t)=-\frac{p(t)}{q(t)}[x'(t)]^2\left(\ln\left(p(t)q(t)\right)\right)', \tag{1.5}$$

whence by (1. 1) we get

$$v' \leqq v\big[(\ln(pq))'\big]_-. \tag{1.6}$$

From (1. 6) and (1. 2) it follows that

$$\ln\frac{v(t)}{v(t_0)} \leqq \int_{t_0}^{t}\big[(\ln(pq))'\big]_- ds \leqq \int_{T}^{\infty}\big[(\ln(pq))'\big]_- ds = C_1 < \infty,$$

which implies

$$v(t) \leqq v(t_0)\exp(C_1) = C_2(t_0) < \infty, \tag{1.7}$$

i.e. $v(t)$ is bounded on $[t_0, T_1)$.

Since $v(t) \geqq 0$, the estimate

$$\int_{t_0}^{T_1}[v']_- ds \leqq v(t_0) + \int_{t_0}^{T_1}[v']_+ ds$$

holds, therefore, by virtue of (1. 6), we have

$$\int_{t_0}^{T_1}|v'|\,ds = \int_{t_0}^{T_1}([v']_+ + [v']_-)\,ds \leqq v(t_0) + 2\int_{t_0}^{T_1}[v']_+ ds \leqq \tag{1.8}$$

$$\leqq C_2(t_0) + 2C_2(t_0)\int_{T}^{\infty}\big[(\ln(pq))'\big]_- ds = C_2(t_0)(1+2C_1) < \infty,$$

i.e. $v(t)$ is of bounded variation on $[t_0, T_1)$, and consequently $v(t)$ tends to a finite limit as $t \to T_1 - 0$; $x(t)$ and $x'(t)$ are bounded on every finite subinterval of $[t_0, T_1)$.

Applying Lemma 1 of [1], we get $T_1 = \infty$, i.e. a) is true. Then (1. 8) implies b), from which it is easy to see, that also c) is true.

Lemma 1. 2. *Any solution $x(t)$ of* (E) *is either oscillatory or monotonic on an appropriate interval* $[T_0, \infty)$.

Proof. The zero solution of (E) obviously satisfies the statement of the lemma. Suppose now that $x(t) \not\equiv 0$. Then, by virtue of the uniqueness of the zero solution and the special type of (E), $x(t)$ and $x'(t)$ have only zeros of multiplicity one and these zeros constitute a discrete set in every finite interval. Therefore it is sufficient to prove that between any two consecutive zeros of $x'(t)$ there is one and only one zero of $x(t)$.

Let t', t'' be two consecutive zeros of $x'(t)$. Integrating (E) from t' to t'' we obtain

$$\int_{t'}^{t''} q(s) f(x(s))\,ds = -[p(s)x'(s)]_{t'}^{t''} = 0.$$

But $q(t) > 0$, so $f(x(s))$ changes its sign in (t', t''), consequently, because of (A_2), $x(t)$ also has to change its sign in the same interval. Therefore $x(t)$ vanishes at some point of (t', t''). On the other hand, since t', t'' are consecutive zeros of $x'(t)$, the solution $x(t)$ vanishes at most once in (t', t''), which completes the proof.

2. Non-oscillatory solutions

Theorem 2. 1. *If* (1. 2) *holds and every solution of* (E) *satisfies* (R_1), *then*

$$\int_T^\infty \frac{1}{p(t)} \int_T^t q(s)\,ds\,dt = \infty. \tag{2. 1}$$

Proof. Suppose the contrary, i.e.

$$\int_T^\infty \frac{1}{p(t)} \int_T^t q(s)\,ds\,dt < \infty. \tag{2. 2}$$

Then we shall prove that there exists a solution $x_0(t)$ of (E) with

$$\liminf_{t\to\infty} |x_0(t)| > 0. \tag{2. 3}$$

Integrating (E) from $\hat{t}$ to t we obtain

$$x'(t) = \frac{1}{p(t)} p(\hat{t})x'(\hat{t}) - \frac{1}{p(t)} \int_{\hat{t}}^t q(s)f(x(s))\,ds, \tag{2. 4}$$

where $\hat{t}$ is an arbitrary fixed point of $[T, \infty)$. Therefore the solution $\hat{x}(t)=x(t; \hat{t}, 1, 0)$ of (E) satisfies the identity

$$\hat{x}(t) = 1 - \int_{\hat{t}}^t \frac{1}{p(s)} \int_{\hat{t}}^s q(\tau)f(\hat{x}(\tau))\,d\tau\,ds. \tag{2. 5}$$

Using (1. 7) for the function $v(t)=V(t, \hat{x}(t), \hat{x}'(t))$ we get $2F(\hat{x}(t)) \leqq v(t) \leqq 2F(1)\exp(C_1)$, from which, by virtue of (A_2), it follows that there exists a constant C such that for any $\hat{t}$ $|f(\hat{x}(t))|<C$ holds on $[\hat{t}, \infty)$, (C is independent of $\hat{t}$). On the other hand, in view of (2. 2) and $q \geqq 0$, there exists a number $t_0 \geqq T$ such that

$$\int_{t_0}^t \frac{1}{p(s)} \int_{t_0}^s q(\tau)\,d\tau\,ds < \frac{1}{2C}, \qquad (t \geqq t_0). \tag{2. 6}$$

Set $x_0(t)=x(t; t_0, 1, 0)$, then (2. 5), (2. 6) and $|f(x_0(t))<C|$ imply the estimation

$$|x_0(t)| \geqq 1 - C\int_{t_0}^t \frac{1}{p(s)} \int_{t_0}^s q(\tau)\,d\tau\,ds \geqq 1 - \frac{1}{2} = \frac{1}{2}, \qquad (t \geqq t_0),$$

i.e. the solution $x_0(t)$ has property (2. 3), which contradicts (R_1), therefore (2. 2) is false. The theorem is proved.

Remark 2. 1. In the preceding proof it has been also shown that (1. 2) and (2. 2) imply (E) to have a non-oscillatory solution. (By virtue of (2. 3) $x_0(t)$ is non-oscillatory.)

Theorem 2. 2. (1. 2) *and* (2. 1) *imply that every non-oscillatory solution of* (E) *satisfies* (R_1).

To prove this theorem we need the following

Proposition 2. 1. *If the function $h(t)\in C'[T,\infty)$ is positive and satisfies*

$$\int_T^\infty [(\ln h(t))']_-\,dt<\infty,$$

then $\liminf_{t\to\infty} h(t)>0$.

PROOF of Proposition 2. 1. Suppose that the statement is false. Then there exists a sequence $\{t_n\}_{n=1}^\infty$ such that

$$t_n\geqq T,\quad \lim_{n\to\infty} t_n=\infty,\quad \lim_{n\to\infty} h(t_n)=0. \tag{2. 7}$$

Since for arbitrary real number a the identity $[a]_-=\frac{1}{2}(|a|-a)$ holds, we have

$$\int_T^{t_n}[(\ln h(t))']_-\,dt\geqq -\frac{1}{2}\int_T^{t_n}(\ln h(t))'\,dt=\frac{1}{2}\ln\frac{h(T)}{h(t_n)} \tag{2. 8}$$

for every n. Applying (2. 7) we see that (2. 8) contradicts our assumptions on $h(t)$. This concludes the proof.

PROOF of Theorem 2. 2. Let $x(t)$ be a non-oscillatory solution of (E). In view of Lemma 1. 1 and 1. 2 $x(t)$ is monotonic and bounded for t large enough, consequently $x(t)\to v$ as $t\to\infty$. We are going prove that $v=0$.

Integrating twice (E) we obtain the identity

$$x(t)=x(t_0)+p(t_0)x'(t_0)\int_{t_0}^t\frac{1}{p(s)}\,ds-\int_{t_0}^t\frac{1}{p(s)}\int_{t_0}^s q(\tau)f(x(\tau))\,d\tau\,ds. \tag{2. 9}$$

Suppose $v>0$. Then there exists a $T_0\geqq t_0$ such that $\frac{1}{2}v<x(t)<\frac{3}{2}v$ provided $t\geqq T_0$. By virtue of (A_2) we have

$$k=\inf_{\frac{1}{2}v<u<\frac{3}{2}v}\{f(u)\}>0.$$

From (2. 9) it follows that

$$x(t)\leqq x(t_0)+p(t_0)|x'(t_0)|\int_{t_0}^t\frac{1}{p(s)}\,ds-k\int_{t_0}^t\frac{1}{p(s)}\int_{t_0}^s q(\tau)\,d\tau\,ds \tag{2. 10}$$

is valid for all $t\geqq T_0$.

Now we distinguish two cases

a) $\displaystyle\int_{t_0}^\infty\frac{1}{p(s)}\,ds<\infty,$

b) $\displaystyle\int_{t_0}^\infty\frac{1}{p(s)}\,ds=\infty.$

Ad a) Using assumption (2. 1), from (2. 10) we get $x(t)\to-\infty$ as $t\to\infty$, which contradicts the boundedness of $x(t)$.

Ad b) By Proposition 2. 1 $p(t)q(t)>c>0$ follows for t large enough, i.e. $q(t)>$ $>c[p(t)]^{-1}$, consequently $\int_{t_0}^{\infty} q(s)\,ds=\infty$. Then t is easy to see (e.g. by L'Hospital's rule) that

$$\lim_{t\to\infty}\left(K\int_{t_0}^{t}\frac{1}{p(s)}\,ds-\int_{t_0}^{t}\frac{1}{p(s)}\int_{t_0}^{s}q(\tau)\,d\tau\,ds\right)=-\infty$$

holds for every constant K. Hence (2. 10) contradicts the boundedness of $x(t)$ in this case too.

Similarly it can be proved that the assumption $v<0$ also contradicts the boundedness of $x(t)$.

So $v\neq 0$ led to a contradiction, and this proves the theorem.

Remark 2. 2. Assumptions (1. 2) and (2. 1) permit (E) to have non-oscillatory solution. This is shown by the following obvious example.

Consider the equation

$$x''+ax'+bx=0, \tag{2. 11}$$

where a, b are positive constants. This equation can be written also into the form

$$(\exp(at)x')'+b\exp(at)x=0.$$

For arbitrary a, b we have $(b\exp(2at))'[b\exp(2at)]^{-1}=2a>0$ and

$$\int_{T}^{\infty}\exp(-at)\int_{T}^{t}b\exp(as)\,ds\,dt=\frac{b}{a}\int_{T}^{\infty}[1-\exp(a(T-t))]\,dt=\infty,$$

i.e. (1. 2) and (2. 1) are satisfied. On the other hand, $a>2b$ implies for every solution of (2. 11) to be oscillatory, $a\leqq 2b$ however implies for every solution of (2. 11) to be non-oscillatory.

3. Oscillatory solutions

In the present Sec. we suppose that $q(t)\in C^2[T,\infty)$,

$$(p(t)q(t))'\geqq 0 \tag{3. 1}$$

for $t\geqq T$, furthermore there exists a positive number γ such that

$$\gamma xf(x)\geqq 2F(x), \qquad (-\infty<x<\infty). \tag{3. 2}$$

Theorem 3. 1. *If there exists a positive function $d(t)\in C^3[T,\infty)$ such that*

$$d'(t)>0,\quad \lim_{t\to\infty}d(t)=\infty, \tag{3. 3}$$

$$\mu=\liminf_{t\to\infty}\frac{[\ln(p(t)q(t))]'}{[\ln d(t)]'}>\gamma \tag{3. 4}$$

and

$$\int_{T}^{t}\left[\left(\left(\frac{d'(s)}{q(s)}\right)'p(s)\right)'\right]_{-}ds=o(d(t)),\qquad (t\to\infty), \tag{3. 5}$$

then every oscillatory solution $x(t)$ of (E) *satisfies the relations* (R_1) *and*

$$(R_2) \qquad \lim_{t\to\infty}\left[\frac{p(t)}{q(t)}\right]^{1/2} x'(t) = 0.$$

PROOF. Let $x(t)$ be an oscillatory solution of (E) and consider the positive function $v(t)=V(t, x(t), x'(t))$. In view of (3. 1), it follows from (1. 5) that $v(t)$ decreases, consequently it tends to a finite limit λ as $t\to\infty$ and $\lambda\geqq 0$. The theorem will be proved if we show that $\lambda=0$.

Suppose $\lambda>0$. Then for an arbitrary $\varepsilon>0$ there exists a $T_1=T_1(\varepsilon)$ such that if $t\geqq T_1$ then

$$(3.6) \qquad \lambda \leqq v(t) \leqq (1+\varepsilon)\lambda.$$

Further, by virtue of assumption (3. 4) there exists a T_2 such that

$$(3.7) \qquad \frac{d(t)}{d'(t)}\frac{(p(t)q(t))'}{p(t)q(t)} > \frac{\gamma+\mu}{2} > \gamma$$

provided $t\geqq T_2$. Let $T_3=\max\{T_1, T_2\}$.

$x(t)$ being oscillatory, there exists a sequence $\{t_n\}_{n=1}^{\infty}$ having the properties

$$(3.8) \qquad t_1 \geqq T_3, \quad x(t_n)=0, \qquad (n=1,2,3,\ldots), \qquad \lim_{n\to\infty} t_n = \infty.$$

Now introduce the following notation

$$(3.9) \qquad w = dv+\gamma\frac{d'}{q}pxx'-\frac{\gamma}{2}\left(\frac{d'}{q}\right)'px^2.$$

Applying (E), a simple differentiation shows that

$$w' = \frac{p}{q}(x')^2\left[(1+\gamma)d'-d\frac{(pq)'}{pq}\right]-\frac{\gamma}{2}\left(\left(\frac{d'}{q}\right)'p\right)'x^2+d'\big(2F(x)-\gamma xf(x)\big).$$

Integrating this inequality from T_3 to t_n and using (1. 1), (3. 2), (3. 8) and (3. 9) we have

$$d(t_n)v(t_n) \leqq O(1)+\int_{T_3}^{t_n} d'\left[1+\gamma-\frac{d}{d'}\frac{(pq)'}{pq}\right]_+ v\,dt+\frac{\gamma}{2}\int_{T_3}^{t_n}\left[\left(\left(\frac{d'}{q}\right)'p\right)'\right]_- x^2\,dt, \qquad (n\to\infty),$$

whence, by virtue of (3. 5), (3. 6), (3. 7) and the boundedness of $x(t)$, it can be obtained that

$$(3.10) \quad \lambda d(t_n) \leqq O(1)+\left[1+\gamma-\frac{\gamma+\mu}{2}\right]_+(1+\varepsilon)\lambda d(t_n)+K\frac{\gamma}{2}\,o\big(d(t_n)\big), \qquad (n\to\infty),$$

where K is a constant such that $x^2<K$. Dividing both sides of the inequality (3. 10) by $\lambda d(t_n)$ and letting $n\to\infty$, in view of (3. 3) we have the estimation

$$1 \leqq \left[1-\frac{\mu-\gamma}{2}\right]_+(1+\varepsilon).$$

This yields the desired contradiction with arbitrary $\varepsilon>0$ if $\mu-\gamma \geqq 2$, and with $\varepsilon < (\mu-\gamma)[2-(\mu-\gamma)]^{-1}$ if $\mu-\gamma < 2$.

Therefore $\lambda=0$, and this proves the theorem.

The following corollaries illustrate the scope of Theorem 3. 1. They are obtained from this one by taking $d(t)\equiv t$, $d(t) \equiv \int_T^t p^\alpha q\,ds$ and $d(t)\equiv \int_T^t q\,ds$ respectively.

Corollary 3. 1. *If*

$$\liminf_{t\to\infty}\left[t(\ln(p(t)q(t)))'\right] > \gamma \tag{3.11}$$

and

$$\int_T^t \left[\left(\frac{q'}{q}\frac{p}{q}\right)'\right]_+ ds = o(t), \qquad (t\to\infty),$$

then every oscillatory solution satisfies (R_1) *and* (R_2).

Remark 3. 1. D. Willett and J. S. W. Wong have obtained a result ([1], Corollary 2. 1) according to which (3. 10) already implies (R_1) and (R_2), but that result is not correct.

Corollary 3. 2. *If there exists a positive number* α *such that*

$$\int_T^\infty p^\alpha q\,dt = \infty, \quad \liminf_{t\to\infty}\frac{[\ln(p(t)q(t))]'}{\left[\ln\int_T^t p^\alpha q\,ds\right]'} > \gamma$$

and $p^{\alpha+1}$ *is convex, then every oscillatory solution of* (E) *satisfies* (R_1) *and* (R_2).

Corollary 3. 3. *If*

$$\int_T^\infty q\,dt = \infty, \quad \liminf_{t\to\infty}\left(\frac{(p(t)q(t))'}{p(t)q^2(t)}\int_T^t q\,ds\right) > \gamma,$$

then every oscillatory solution of (E) *satisfies* (R_1) *and* (R_2).

4. Global asymptotic stability of the zero solution

In this Sec. we suppose that

$$\liminf_{t\to\infty}\frac{p(t)}{q(t)} > 0. \tag{4.1}$$

So for an arbitrary solution $x(t)$ of (E) relation (S) is implied by $v(t)\to 0$ as $t\to\infty$.

Theorem 4. 1. *If there exists a solution* $\hat{x}(t)\not\equiv 0$ *of* (E) *such that* $V(t, \hat{x}(t), \hat{x}'(t))= =v(t)\to 0$ *as* $t\to\infty$, *then*

$$\int_T^\infty [(\ln(p(t)q(t)))']_+\,dt = \infty.$$

PROOF. From (1. 1) and (1. 5) it follows that

$$\frac{v'}{v} = -\frac{1}{v}\frac{p}{q}(\hat{x}')^2(\ln(pq))' \geqq -[(\ln(pq))']_+.$$

Sice $v(t)\to 0$ as $t\to\infty$ we have

$$\int_T^t [(\ln(pq))']_+\,ds \geqq -\int_T^t \frac{v'}{v}\,ds = \ln\frac{v(T)}{v(t)} \to \infty$$

as $t\to\infty$, which was to be proved.

Theorem 4. 2. *Suppose that* (1. 2), (2. 1) *and* (4. 1) *are satisfied. If*

$$\int_S [(\ln(p(t)q(t)))']_+\,dt = \infty \tag{4. 2}$$

holds on every set $S = \bigcup_{n=1}^{\infty}(a_n, b_n)$ *such that*

$$T \leqq a_1,\quad a_n < b_n < a_{n+1},\quad b_n - a_n \geqq \delta > 0, \qquad (n=1, 2, 3, \ldots),$$

then the zero solution of (E) *is g. a. s..*

Remark 4. 1. If $(\ln(p(t)q(t)))'$ is non-negative, periodic and does not vanish identically on any subinterval of $[T, \infty)$, then (4. 2) is obviously satisfied.

It is easy to prove that (4. 2) and the following statement are equivalent: for every $\delta > 0$

$$\liminf_{t\to\infty} \int_t^{t+\delta} [(\ln(p(s)q(s)))']_+\,ds > 0$$

is valid.

PROOF. Let $x(t)$ be a solution of (E). In view of Lemma 1. 1 $x(t)$ exists in $[T, \infty)$, $x(t)$ and $u(t) = p(t)[q(t)]^{-1}[x'(t)]^2$ are bounded.

First, we shall prove that $u(t)$ tends to 0 as $t\to\infty$.

Suppose the contrary, i.e.

$$\limsup_{t\to\infty} u(t) = \lambda > 0, \tag{4. 3}$$

and consider the open unbounded set

$$H = \left\{t : t \geqq T,\ u(t) > \frac{\lambda}{3}\right\}. \tag{4. 4}$$

By virtue of Lemma 1. 1 $v(t)$ is of bounded variation on $[T, \infty)$, consequently, applying (1. 5) and (4. 4), we have

$$\infty > \int_0^{\infty} |v'|\,dt \geqq \int_H u\,|(\ln(pq))'|\,dt \geqq \frac{\lambda}{3}\int_H |(\ln(pq))'|\,dt, \tag{4. 5}$$

and therefore, as a consequence of (4. 2), H does not contain an interval of the type (ξ, ∞), and hence

$$\liminf_{t\to\infty} u(t) \leq \frac{\lambda}{3}. \tag{4. 6}$$

(4. 3) and (4. 6) imply that there exists a sequence of intervals $\omega_n=(t_n', t_n'')\subset H$, $(n=1, 2, 3, \ldots)$ such that

$$t_n'<t_n''<t_{n+1}',\ u(t_n') = u(t_n'') = \frac{\lambda}{3}, \qquad (n=1, 2, 3, \ldots), \qquad \lim_{n\to\infty} t_n' = \infty, \tag{4. 7}$$

and for every n there exists a $\tau_n\in\omega_n$ with

$$u(\tau_n) = \frac{2}{3}\lambda. \tag{4. 8}$$

From (4. 5) and (4. 7), by assumption (4. 2), we have

$$\liminf_{n\to\infty} m(\omega_n) = 0, \quad \lim_{n\to\infty} \int_{\omega_n} u\left|(\ln(pq))'\right| dt = 0. \tag{4. 9}$$

Since $u' = v'-2[F(x)]'$, (4. 7) and (4. 8) imply

$$\frac{\lambda}{3} \leq \int_{\omega_n} |u'|\, dt \leq \int_{\omega_n} u\left|(\ln(pq))'\right| dt + 2\int_{\omega_n} |x'f(x)|\, dt \tag{4. 10}$$

for every n. $|x'f(x)|$ is bounded because $x(t)$ and $u(t)$ are bounded and (4. 1) holds. Therefore from (4. 10), by virtue of (4. 9), we obtain the estimate $\lambda\leq o(1)$ as $n\to\infty$, which contradicts the fact that $\lambda>0$, consequently $u(t)\to 0$ as $t\to 0$. Then, using (4. 1), we have

$$\lim_{t\to\infty} x'(t) = 0. \tag{4. 11}$$

It remains to verify $x(t)\to 0$ as $t\to\infty$.

Lemma 1. 1 assures for $v(t)$ to tend to a finite limit, hence from (5. 4) the same property follows for $F(x(t))$. Consequently, taking assumption (A_2) into consideration, it is easy to see that $\lim_{t\to\infty} x(t)=v$ exists too. If $x(t)$ is oscillatory, then obviously $v=0$. On the other hand, by Theorem 2. 2 $v=0$ follows for non-oscillatory solutions too. This completes the proof.

5. Applications and examples

I. Consider the equation

$$x''+a(t)x'+b(t)f(x) = 0, \tag{E'}$$

where $a(t)\in C[T, \infty)$, $b(t)\in C'[T, \infty)$; $b(t)>0$ on $[T, \infty)$ and $f(x)$ satisfies (A_2).

Using the notation

$$A(t) = \exp\left(\int_T^t a(s)\, ds\right) \tag{5. 1}$$

(E′) can be written in the form

$$(A(t)x')' + b(t)A(t)f(x) = 0, \tag{5.2}$$

which is a special case of the equation (E), namely

$$p(t) \equiv A(t), \quad q(t) \equiv b(t)A(t). \tag{5.3}$$

So equation (5. 2) satisfies assumptions (A_1) and (A_2); suppose that (A_3) is satisfied, too. Consequently our results are applicable for (E′). To illustrate this we establish the following corollaries of Theorems 3. 4 and 4. 2.

Corollary 5. 1. *Suppose that* $b(t) \in C^2[T, \infty)$,

$$2a(t) + \ln (b(t))' \geqq 0, \qquad (t \geqq T), \tag{5.4}$$

and (3. 2) *is satisfied. If there exists a function* $d(t) \in C^3[T, \infty)$ *with* (3. 3);

$$\liminf_{t\to\infty} \frac{2a(t) + (\ln b(t))'}{(\ln d(t))'} > \gamma$$

and

$$\int_T^t \left[\left(\left(\frac{d'}{b}\right)' - d' \frac{a}{b}\right)'\right]_- ds = o(d(t)), \qquad (t \to \infty),$$

then for every oscillatory solution $x(t)$ *of* (E′)

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{1}{[b(t)]^{1/2}} x'(t) = 0 \tag{R'}$$

is valid.

Corollary 5. 2. *Suppose that* $b(t)$ *is bounded and* (5. 4) *is satisfied. If*

$$\int_T^\infty \frac{1}{A(t)} \int_T^t b(s)A(s)\,ds\,dt = \infty \tag{5.5}$$

and for every $\delta > 0$

$$\liminf_{t\to\infty} \left[\int_t^{t+\delta} a(s)\,ds + \ln \frac{b(t+\delta)}{b(t)}\right] > 0, \tag{5.6}$$

then the zero solution of (E′) *is g. a. s..*

Proof. By virtue of (5. 3) $(\ln(pq))' = 2a + (\ln b)'$, therefore, applying Remark 4. 1, the statements follow obviously from the two theorems mentioned above.

The following two corollaries illustrate the scope of Corollary 5. 1. They are obtained from this one by taking $d(t) \equiv \int_T^t b(s)ds$ and $d(t) \equiv \int_T^t b(s)[a(s)]^{-1}ds$, respectively.

Corollary 5. 3. *Suppose that* (3. 2) *and* (5. 4) *are satisfied and* $a(t) \in C'[T, \infty)$. *If*

$$\int_T^\infty b(t)\,dt = \infty, \quad \liminf_{t\to\infty} \left[\int_T^t b(s)\,ds \left(2\frac{a(t)}{b(t)} + \frac{b'(t)}{b^2(t)}\right)\right] > \gamma$$

and

$$\int_T^t [a']_+ \, ds = o\Big(\int_T^t b(s)\, ds\Big), \qquad (t \to \infty),$$

then (R′) *holds for every oscillatory solution of* (E′).

Corollary 5. 4. *Suppose that* (3. 2) *and* (5. 4) *are satisfied, and* $a(t) \in C^2[T, \infty)$. *If*

$$\int_T^\infty \frac{b(t)}{a(t)}\, dt = \infty, \quad \liminf_{t\to\infty} \left[\int_T^t \frac{b(s)}{a(s)}\, ds \left(\frac{2a^2(t)}{b(t)} + \frac{a(t)b'(t)}{b^2(t)}\right)\right] > \gamma$$

and

$$\int_T^t \left[\left(\frac{1}{a}\right)''\right]_- ds = o\left(\int_T^t \frac{b(s)}{a(s)}\, ds\right), \qquad (t \to \infty),$$

then (R′) *is valid for every oscillatory solution of* (E′).

R. A. Smith [4] has given the following two results concerning the linear equation

(LE′) $$x'' + a(t)x' + x = 0.$$

A) If (5. 5) is satisfied and $a(t) \geqq \varepsilon > 0$, then the zero solution of (LE′) is g. a. s..

B) If $a(t)$ is positive, decreasing and $\int_0^\infty a(t)\,dt = \infty$, then the zero solution of (LE′) is g. a. s..

Corollary 5. 2 is a generalized and sharpened form of A). Using our results a generalization of a somewhat weakened form of B) can be given.

Theorem 5. 1. *If* (3. 2) *is satisfied;* $a(t)$ *is positive, decreasing and*

(5. 7) $$\lim_{t\to\infty} ta(t) > \frac{\gamma}{2},$$

then the zero solution of the equation

$$x'' + a(t)x' + f(x) = 0$$

is g. a. s..

Proof. The assumptions of Corollary 5. 3 are satisfied, so (R′) is valid for every oscillatory solution. Then, because of $b(t) \equiv 1$, these satisfy (S), too. So it is sufficient to consider the non-oscillatory solutions.

First, we shall prove that (5. 5) holds. In view of (5. 7) for t large enough $a(t) > \gamma[2t]^{-1}$, therefore $A(t) \to \infty$ as $t \to \infty$. By L'Hospital's rule we have

$$\lim_{t\to\infty} \frac{1}{A(t)} \int_T^t A(s)\, ds = \lim_{t\to\infty} \frac{1}{a(t)} > 0,$$

i.e. (5. 5) is valid.

Now, as a consequence of Theorem 2. 2, (R_1) holds for every non-oscillatory solution $x(t)$. It remains to verify that $x'(t) \to 0$ as $t \to \infty$. This can be obtained combining identity (2. 4) by relations (R_1) and $q(t) \equiv p(t) \equiv A(t) \to \infty$ as $t \to \infty$.

The proof is complete.

II. The following two examples show the relation of our three theorems to one another and to the theorem of D. Willett and J. S. W. Wong ([1], Theorem 1. 1).

Example 5. 1. Set $p(t)\equiv t^{3/2}$, $q(t)\equiv t^{-1/2}$, and let $f(x)$ be a function satisfying (A_2) and (3. 2). Then $p(t)q(t)\equiv t$, so it is easy to verify that

(i) Theorem 4. 2 is not applicable;
(ii) Theorem 2. 2 is applicable;
(iii) Theorem 3. 1 is applicable taking $d(t)\equiv t$;
(iv) Willet's and Wong's theorem is not applicable.

Example 5. 2. Set $p(t)\equiv \exp[\frac{1}{4}(t-\sin 2t)]$, $q(t)\equiv \exp(\frac{1}{4}t)$, and let $f(x)$ be a function satisfying (A_2) and (3. 2). Then $(\ln(pq))'=\sin^2(t)$, so it is easy to verify that

(i) Theorem 4. 2 is applicable;
(ii) Theorem 3. 1 is not applicable;
(iii) Willett's and Wong's theorem is not applicable.

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