Krabbe's generalized functions as convolutions quotients

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In this paper applying the methods of [2] and [7] Krabbe's generalized functions (see [1]) will be embedded in a quotient ring which will be used for consideration of that.

A convolution algebra of locally integrable functions

Definition 1. Let $\mathcal{I}=\langle\alpha, \beta\rangle$ be an arbitrary (open, half-open or closed, bounded or unbounded) interval of the real line with endpoints α and β such that $0\in\mathcal{I}$.

Let $\mathcal{L}(\mathcal{I})$ denote the vector space of locally Lebesgue-integrable complex-valued function on $\mathcal{I}$ over the complex number field $\mathcal{K}$.

For $f, g\in\mathcal{L}(\mathcal{I})$, let

$$(fg)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau \qquad (t\in\mathcal{I}). \tag{1}$$

It is easily seen that $\mathcal{L}(\mathcal{I})$ with (1) is a commutative algebra over $\mathcal{K}$. Moreover it follows immediately from Titchmarsh's theorem (see [3]) that $f\in\mathcal{L}(\mathcal{I})$ ($f(t)\neq 0$ almost everywhere on $\mathcal{I}$) is a divisor of zero in $\mathcal{L}(\mathcal{I})$ if and only if there is a real number ϑ such that $0<\vartheta<1$ and $f(t)=0$ almost everywhere on at least one of the sets $\vartheta\alpha<t\leqq 0$ and $0\leqq t<\vartheta\beta$ which is not empty.

Definition 2. Let $\mathcal{T}(\mathcal{I})$ be the subset of $\mathcal{L}(\mathcal{I})$ to which all those functions w belong which are infinitely differentiable in $\mathcal{I}$ and satisfy $w^{(k)}(0)=0$ for every integer $k\geqq 0$.

Obviously $\mathcal{T}(\mathcal{I})$ is a subalgebra of $\mathcal{L}(\mathcal{I})$.

Convolution quotients

In theory of algebras it is known that every commutative algebra A over a field K with at least one element not a divisor of zero has a quotient ring Q which is a commutative ring with unit element. Furthermore K and A are embedded in Q as follows:

$$\alpha \stackrel{\text{def}}{=} \frac{\alpha b}{b} \qquad (\alpha \in K), \tag{2}$$

$$a \stackrel{\text{def}}{=} \frac{ab}{b} \qquad (a \in A), \tag{3}$$

where $b \in A$ is not a divisor of zero in A.

Definition 3. Let $Q(\mathcal{I})$ be the quotient ring of $\mathcal{L}(\mathcal{I})$.

Remark 1. It is obvious that $Q([0, \infty))$ is the Mikusiński operator field (see [3]) and $Q([0, T])$ where $0 < T < \infty$ is the Mikusiński ring (see [4]).

It is easy to show that $Q(\langle\alpha, \beta\rangle)$ is the direct sum of $Q(\langle\alpha, 0])$ and $Q([0, \beta\rangle)$ if $\alpha < 0 < \beta$, and $Q(\langle\alpha, 0])$ and $Q([0, -\alpha\rangle)$ are isomorphic. In [5] it is proved that $Q([0, T))$ and $Q([0, T])$ are also isomorphic.

In this way the investigation of $Q(\mathcal{I})$ can be reduced to that of $Q([0, T])$ and $Q([0, \infty))$. In [6] an operational calculus belonging to $Q((-\infty, \infty))$ has been constructed in the preceding way.

Theorem 1. *For each $q \in Q(\mathcal{I})$ there exist $w, \omega \in \mathcal{T}(\mathcal{I})$ such that $q = \frac{w}{\omega}$.*

Proof. Let τ be an element of $\mathcal{T}(\mathcal{I})$ which is not a divisor of zero in $\mathcal{L}(\mathcal{I})$. If $q = \frac{f}{g} \in Q(\mathcal{I})$ then $w = f\tau$ and $\omega = g\tau$ have the required properties.

Corollary 1.1. The quotient rings of $\mathcal{T}(\mathcal{I})$ and $\mathcal{L}(\mathcal{I})$ are isomorphic.

Definition 4. Let $\{\alpha\}$ denote the constant function of $\mathcal{L}(\mathcal{I})$ whose value is α.

Theorem 2. *If $f \in \mathcal{L}(\mathcal{I})$ is locally absolutely continuous then for the convolution quotient $s = \frac{1}{\{1\}}$*

$$sf = f' + f(0) \tag{4}$$

holds.

This formula is a generalization of the well-known formulas of Mikusiński (see [3] p. 349 and [4]), and can be proved quite similarly.

Generalized functions

Definition 5. Let $\mathcal{G}(\mathcal{I})$ be the set of all mappings F which map $\mathcal{T}(\mathcal{I})$ into $\mathcal{T}(\mathcal{I})$ such that

$$F(w\omega) = F(w)\omega \tag{5}$$

holds for all $w, \omega \in \mathcal{T}(\mathcal{I})$.

In $\mathcal{G}(\mathcal{I})$ let the usual equality, addition and composition of mappings be defined.

It will be shown that $\mathcal{G}(\mathcal{I})$ forms a commutative ring with unit element under the preceding operations.

Remark 2. By terms of [2] the elements of $\mathcal{G}(\mathcal{I})$ is maximal multiplier operators of $\mathcal{T}(\mathcal{I})$. Moreover, it is proved that the quotient algebra of a commutative algebra A (if it exists) is isomorphic to a subalgebra of the algebra of all maximal multipliers of A whose domain is without annihilator.

Theorem 3. *Let ω_0 be a fixed element of $\mathcal{T}(\mathcal{I})$ which is not a divisor of zero in $\mathcal{T}(\mathcal{I})$. If $F\in\mathcal{G}(\mathcal{I})$, then*

$$F(w)=\frac{F(\omega_0)}{\omega_0}w \qquad (w\in\mathcal{T}(\mathcal{I})). \tag{6}$$

PROOF. Let $F\in\mathcal{G}(\mathcal{I})$, then for ω_0 and all $w\in\mathcal{T}(\mathcal{I})$ by (5) we have $F(w)\omega_0=$ $=F(\omega_0)w$. Hence, because ω_0 is not a divisor of zero in $\mathcal{T}(\mathcal{I})$ the formula (6) follows.

Corollary 3. 1. $\mathcal{G}(\mathcal{I})$ is a commutative ring with unit element.

Corollary 3. 2. From (6) it follows that F is linear. Thus $\mathcal{G}(\mathcal{I})$ is the ring of Krabbe's generalized functions if $\mathcal{I}$ is open (see [1]).

Corollary 3. 3. $F\in\mathcal{G}(\mathcal{I})$ has an inverse in $\mathcal{G}(\mathcal{I})$ if and only if the range of F is $\mathcal{T}(\mathcal{I})$, and then

$$F^{-1}(w)=\frac{\omega_0}{F(\omega_0)}w \qquad (w\in\mathcal{T}(\mathcal{I})) \tag{7}$$

is the inverse of F in $\mathcal{G}(\mathcal{I})$.

PROOF. Suppose that $F\in\mathcal{G}(\mathcal{I})$ has an inverse in $\mathcal{G}(\mathcal{I})$, i.e., there exists a $G\in\mathcal{G}(\mathcal{I})$ such that $(FG)(w)=F(G(w))=w$ for all $w\in\mathcal{T}(\mathcal{I})$. Thus rng $F=\mathcal{T}(\mathcal{I})$. On the other hand, then by (6) $F(w)=\frac{F(\omega_0)}{\omega_0}w$ and $G(w)=\frac{G(\omega_0)}{\omega_0}w$, and so $(FG)(w)=$ $=\frac{F(\omega_0)G(\omega_0)}{\omega_0^2}w$ $(w\in\mathcal{T}(\mathcal{I}))$. Hence $F(\omega_0)G(\omega_0)=\omega_0^2$. Since ω_0 is not a divisor of zero in $\mathcal{T}(\mathcal{I})$ it follows that $F(\omega_0)$ is also not a divisor of zero in $\mathcal{T}(\mathcal{I})$ and $\frac{G(\omega_0)}{\omega_0}=\frac{F(\omega_0)}{\omega_0}$.

Conversely, if rng $F=\mathcal{T}(\mathcal{I})$ then there exists at least one $w_0\in\mathcal{T}(\mathcal{I})$ such that $\omega_0=F(w_0)$. Thus by (6) we have $\omega_0=F(w_0)=\frac{F(\omega_0)}{\omega_0}w_0$ and so $F(\omega_0)w_0=w_0^2$. Since ω_0 is not a divisor of zero in $\mathcal{T}(\mathcal{I})$, it follows that $F(\omega_0)$ is also not a divisor of zero in $\mathcal{T}(\mathcal{I})$. Now it is clear that $G(w)=\frac{\omega_0}{F(\omega_0)}w$ $(w\in\mathcal{T}(\mathcal{I}))$ is the inverse of F in $\mathcal{G}(\mathcal{I})$.

Theorem 4.

$$F\overset{\text{def}}{=}\frac{F(\omega_0)}{\omega_0} \qquad (F\in\mathcal{G}(\mathcal{I})) \tag{8}$$

is an embedding of $\mathscr{G}(\mathscr{I})$ in $Q(\mathscr{I})$ and after this embedding

$$\mathscr{G}(\mathscr{I})\subset Q(\mathscr{I}),\ \mathscr{G}(\mathscr{I})\neq Q(\mathscr{I}) \tag{9}$$

holds.

We will show only that $\mathscr{G}(\mathscr{I})\neq Q(\mathscr{I})$. Namely, for example $\frac{1}{\omega_0}\in Q(\mathscr{I})$ but $\frac{1}{\omega_0}\notin\mathscr{G}(\mathscr{I})$. Indeed let us suppose the contrary that $\frac{1}{\omega_0}\in\mathscr{G}(\mathscr{I})$. Then for $F=\frac{1}{\omega_0}$ by (8) $\frac{1}{\omega_0}=\frac{F(\omega_0)}{\omega_0}$ follows, and so $F(\omega_0)=1$ is a contradiction.

Remark 3. In this sense arbitrary convolution quotient q is identifiable with the mapping

$$F_q(w)=qw \qquad \big(w\in\{w\in\mathscr{T}(\mathscr{I})\ :\ qw\in\mathscr{T}(\mathscr{I})\}\big) \tag{10}$$

and so the convolution quotient q is a linear operator which maps an ideal of $\mathscr{T}(\mathscr{I})$ (with at least one element not a divisor of zero in $\mathscr{T}(\mathscr{I})$) into $\mathscr{T}(\mathscr{I})$ satisfying the equation (5) such that it has no proper extension in $\mathscr{T}(\mathscr{I})$.

Now the following statement is quite obvious:

Theorem 5. *A convolution quotient q is an element of $\mathscr{G}(\mathscr{I})$ if and only if $qw\in\mathscr{T}(\mathscr{I})$ for all $w\in\mathscr{T}(\mathscr{I})$.*

Corollary 5. 1. $K\subset\mathscr{G}(\mathscr{I})$.

Corollary 5. 2. $\mathscr{L}(\mathscr{I})\subset\mathscr{G}(\mathscr{I})$.

Corollary 5. 3. $s\in\mathscr{G}(\mathscr{I})$ and thus the formula (4) holds also in $\mathscr{G}(\mathscr{I})$.

Definition 6. Let $\{\mathscr{L}(\mathscr{I}), s\}$ denote the subring of $Q(\mathscr{I})$ generated by $\mathscr{L}(\mathscr{I})$ and s.

Corollary 5. 4. $\{\mathscr{L}(\mathscr{I}), s\}\subset\mathscr{G}(\mathscr{I})$.

Problem. Is the equality $\mathscr{G}(\mathscr{I})=\{\mathscr{L}(\mathscr{I}), s\}$ true?

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