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Weighted Nikolskiĭ-type inequalities II

By I. JOÓ (Budapest)

Abstract. In the paper is proved that the Nikolskiĭ-type inequality (2) below is sharp in the cases: $p \le q$; $p > q, m > 1$; $p > q, 0 < m < q$.

The present paper is a contribution to the investigations initiated by P. Névai, V. Totik and others (see [8], [9] and [1] for further references).

Let

$$w(x) = w_\alpha(x) = |x|^{\alpha/2} \cdot \exp(-|x|^m), \quad x \in \mathbb{R}, \quad m > 0.$$

Given p, q and m such that $0 < p$, $q \le \infty$, $m > 0$ define the Nikolskiĭ constant $N_n = N_n(m, p, q)$, $n = 1, 2, \dots$ by

$$N_n(m,p,q) = \begin{cases} n^{1/m(1/p-1/q)} & \text{if } p \le q, \\ n^{(1-1/m)(1/q-1/p)} & \text{if } p > q \text{ and } m > 1, \\ (\log(n+1))^{1/q-1/p} & \text{if } p > q \text{ and } m = 1, \\ 1 & \text{if } p > q \text{ and } 0 < m < 1. \end{cases} \tag{1}$$

For $0 < p \le \infty$ denote $\|f\|_p$ the expression

$$\|f\|_p = \left(\int |f(t)|^p dt \right)^{1/p}.$$

One of the results proved in [1] is the following.

Theorem ([1]). *Suppose $0 < p$, $q \le \infty$, $\alpha \ge 0$, $m > 0$. Then for any polynomial $p_n \in \Pi_n$ of degree $\le n$ we have*

$$\|p_n w_\alpha\|_p \le c \cdot N_n(m,p,q) \cdot \|p_n w_\alpha\|_q, \tag{2}$$

where $c = c(m, p, q)$ is a positive constant independent of n, p_n.

The aim of the present note is to prove that this theorem is sharp in the cases $p \le q$; $p > q$ and $m > 1$; $p > q$ and $0 < m < 1$. We think that this theorem is sharp also in the case of $p > q$ and $m = 1$, but now we are not able to prove it. Namely, we show that in the mentioned cases there exist $c^* > 0$ and polynomials $\{R_n^*\}_{n=1}^\infty$ with $\deg R_n^* \le n$ such that

$$\|R_n^* w_\alpha\|_p \ge c^* N_n(m,p,q) \cdot \|R_n^* w_\alpha\|_q. \tag{3}$$

for $n = 1, 2, \dots$.

For the proof of (3) we need some lemmas. First we prove estimates for the Christoffel function of $w_\alpha(x)$. For the definition and other results see [2], Ch. 1.

Lemma 1 ([3], p. 338, Lemma (2.2))**.** *Let*

$$v^{(\alpha)}(x) = \begin{cases} |x|^\alpha, & |x| < 1 \\ 0, & |x| \ge 1 \end{cases} \quad \alpha > -1,$$

and denote $\lambda_n(v^{(\alpha)}, \xi)$ the n-th Christoffel function of $v^{(\alpha)}(x)$. Then

$$\lambda(v^{(\alpha)}, \xi) \asymp n \begin{cases} n^{-2}, & 1 - \dfrac{c_4}{n^2} \le \xi^2 \le 1, \\ \dfrac{1}{n}|\xi|^\alpha \cdot (1-\xi^2)^{\frac{1}{2}}, & \dfrac{c_5}{n^2} \le \xi^2 \le 1 - \dfrac{c_4}{n^2}, \\ n^{-\alpha-1}, & \xi^2 \le \dfrac{c_5}{n^2} \end{cases} \tag{4}$$

where c_4, $c_5 \in (0,1)$ are arbitrary fixed numbers and concerns n.

Lemma 2. *Let $w_\alpha(x)$ be the function $w_\alpha(x) = |x|^{\frac{\alpha}{2}} \exp(-|x|^m)$ where $\alpha/2 > -1$, $m > 0$. Then*

$$\lambda_n(w_\alpha, x) \le \tag{5}$$
$$K \exp(-|x|^m) \cdot \begin{cases} |x|^{\frac{\alpha}{2}} \cdot n^{\frac{1}{m}-1}, & c_7 n^{\frac{1}{m}-1} \le |x| \le c_6 n^{\frac{1}{m}}, \\ n^{\left(\frac{\alpha}{2}+1\right)\left(\frac{1}{m}-1\right)}, & |x| \le c_7 n^{\frac{1}{m}-1} \end{cases}$$

further

a) *in the case of $m > 1$:*

$$\lambda_n(w_\alpha, x) \ge K \exp(-|x|^m) \cdot \begin{cases} |x|^{\frac{\alpha}{2}} \cdot n^{\frac{1}{m}-1}, & c_7 n^{\frac{1}{m}-1} \le |x| \le c_6 n^{\frac{1}{m}}, \\ n^{\left(\frac{\alpha}{2}+1\right)\left(\frac{1}{m}-1\right)}, & |x| \le c_7 n^{\frac{1}{m}-1} \end{cases}$$

b) *in case* $m=1$ *we have*

$$\lambda_n(w_\alpha,x)\ge K\exp(-|x|)\cdot\begin{cases}\dfrac{|x|^{\frac{\alpha}{2}}}{\log(n+1)}, & \dfrac{c_8}{\log(n+1)}\le|x|\le c_6n,\\[2mm] [\log(n+1)]^{-\frac{\alpha}{2}-1}, & |x|\le\dfrac{c_8}{\log(n+1)}.\end{cases}$$

c) *finally, in case* $0<m<1$*:*

$$\lambda_n(w_\alpha,x)\ge$$

$$K\exp(-|x|^m)\cdot\begin{cases}\dfrac{|x|^{\frac{\alpha}{2}+1-m}}{\log(n+1)}, & \dfrac{c_9}{(\log(n+1))^{\frac{1}{m}}}\le|x|\le c_6n^{\frac{1}{m}},\\[2mm] [\log(n+1)]^{-\frac{\alpha}{2m}-\frac{1}{m}}, & |x|\le\dfrac{c_9}{(\log(n+1))^{\frac{1}{m}}}.\end{cases}$$

Proof. 1. First we prove the upper estimate. We need a suitable polynominal $P_{[n/2]}(x)\in\Pi_{[n/2]}$ which satisfies the following condition:

$$0<c_3\le P^2_{[n/2]}(x)\exp(-|x|^m)\le c_3,\quad |x|\le c_5n^{\frac{1}{m}}. \tag{6}$$

We will obtain the desired polynominal using Lubinsky's function G defined by

$$G(x)=1+\sum_{k=1}^{\infty}\left(\frac{em}{2k}\right)^{\frac{2k}{m}}\cdot\frac{1}{\sqrt{k}}\cdot x^{2k} \tag{7}$$

([4], (17)) which originates from a function introduced by Mittag–Leffler cf. [5].

According to [4] Theorem 6 we have $G(x)\asymp\exp(|x|^m)$, $x\in\mathbb{R}$. If we choose r_n to be the $[n/4]$-th partial sum of the power series in (7) then

$$0<r_n(x)\le c\exp(|x|^m),\quad x\in\mathbb{R}. \tag{8}$$

Moreover, examination of the remainder term $G-r_n$ shows that there exists $b_0>0$ (absolute constant) such that $G(x)\le r_n(x)+o(1)$ uniformly for $|x|\le b_0n^{1/m}$, $n=1,2,\dots$ where $\lim\limits_{n\to\infty}o(1)=0$. Hence we have

$$\exp(|x|^m)\le cr_n(x),\quad |x|\le b_0n^{1/m}. \tag{9}$$

Let $P_{[n/2]}(x)$ be

$$P_{[n/2]}(x)=r_n\left(\frac{x}{\sqrt[m]{2}}\right).$$

Then from (8) and (9) we get

$$0 < c\exp(|x|^m) \le P^2_{[n/2]}(x) \le c\exp(|x|^m), \quad |x| \le b_0 \sqrt[m]{2}\, n^{\frac{1}{m}},$$

i.e. $P_{[n/2]}$ satisfies (6). It is well known that

$$\lambda_n(w_\alpha, x) = \min_{\substack{T_{n-1}\in\Pi_{n-1}\\ T_{n-1}(x)=1}} \int_{-\infty}^{\infty} |T_{n-1}(t)|^2 w_\alpha(t)dt. \tag{10}$$

Applying [6], Theorem 4.16.2, we obtain

$$\begin{aligned}\lambda_n(w_\alpha, x) \le \\ \le c \min_{\substack{T_{[n/2]}\in\Pi_{[n/2]}\\ T_{[n/2]}(x)=1}} \int_{-c_1 n^{1/m}}^{-c_1 n^{1/m}} |T_{[n/2]}(t)|^2 \cdot |P_{[n/2]}(t)/P_{[n/2]}(x)|^2 w_\alpha(t)dt.\end{aligned} \tag{11}$$

Thus by (6) we have

$$\frac{\lambda_n(w_\alpha, x)}{\exp(-|x|^m)} \le c \min_{\substack{T_{[n-2]}\in\Pi_{[n/2]}\\ T_{[n/2]}(x)=1}} \int_{-c_1 n^{1/m}}^{c_1 n^{1/m}} |T_{[n/2]}(t)|^2 \cdot |t|^{\frac{\alpha}{2}} dt, \tag{12}$$

where $|x| \le b_0 \sqrt[m]{2} \cdot n^{1/m}$.

By a change of variables $s = \frac{t}{c_1 n^{1/m}}$ we obtain from (12)

$$\begin{aligned}\frac{\lambda n(w_\alpha, x)}{\exp(-|x|^m)} \le c n^{\frac{1}{m}+\frac{\alpha}{2}\cdot\frac{1}{m}} \cdot \lambda_{[n/2]+1}\left(v^{(\alpha/2)}, \frac{x}{c_1 n^{1/m}}\right), \\ |x| \le b_0 \sqrt[m]{2}\, n^{\frac{1}{m}}.\end{aligned} \tag{13}$$

From (13) the desired (5) follows.

2. Now we prove the lower estimate.

a) The case $m > 1$. There exists a polynomial $P_n(x)$ of degree at most n for which

$$P_n^2 \asymp \exp(-|x|^m), \quad |x| \le B n^{\frac{1}{m}},$$

([7], Theorem 1). Using these polynomials we can prove the estimates.

From (10)

$$\lambda_n(w_\alpha, x) \geq k \min_{\substack{T_{n-1}\in\Pi_{n-1}\\ T_{n-1}(x)=1}} \int\limits_{-Bn^{1/m}}^{Bn^{1/m}} |T_{n-1}(t)|^2 w_\alpha(t)dt$$

$$\geq k \exp(|x|^m) \cdot \min_{\substack{T_{n-1}\in\Pi_{n-1}\\ T_{n-1}(x)=1}} \int\limits_{-Bn^{1/m}}^{Bn^{1/m}} |T_{n-1}(t)^2 \cdot |P_n(t)/P_n(x)|^2 \cdot |t|^{\frac{\alpha}{2}}\, dt$$

$$\geq k \exp(|x|^m) \cdot \min_{\substack{T_{2n-1}\in\Pi_{2n-1}\\ T_{2n-1}(x)=1}} \int\limits_{-Bn^{1/m}}^{Bn^{1/m}} |T_{2n-1}(t)|^2 \cdot |t|^{\frac{\alpha}{2}}\, dt$$

and so

$$\frac{\lambda_n(w_\alpha, x)}{\exp(-|x|^m)} \geq kn^{\frac{1}{m}+\frac{\alpha}{2}\cdot\frac{1}{m}} \cdot \lambda_{2n}\left(v^{(\alpha/2)}, \frac{x}{Bn^{1/m}}\right), \quad |x| \leq Bn^{\frac{1}{m}}. \tag{14}$$

Now we prove that Lemma 2, (a) is valid for $|x| \leq cn^{1/m}$ where c is an arbitrary large constant. Since $\lambda_n(w_\alpha, x)$ is a decreasing function of n we have

$$\frac{\lambda_n(w_\alpha, x)}{\exp(-|x|^m)} \geq \frac{\lambda_{kn}(w_\alpha, x)}{\exp(-|x|^m)} \geq kn^{\frac{1}{m}+\frac{\alpha}{2}\cdot\frac{1}{m}} \cdot \lambda_{2kn}\left(v^{(\alpha/2)}, \frac{x}{B(kn)^{1/m}}\right),$$

$|x| \leq B(kn)^{1/m}$, where k is arbitrary fixed integer. According to (4)

$$\lambda_{2kn}\left(v^{(\alpha)}, \xi\right) \asymp \lambda_n\left(v^{(\alpha)}, \xi\right).$$

Consequently Lemma 2 (a) is valid for $|x| \leq cn^{1/m}$.

b) Case $m = 1$. The calculation is the same as in a) but we use $P_{Ln[\log(n+1)]}(x)$ instead of $P_n(x)$. Here $P_{Ln[\log(n+1)]}(x)$ is of degree at most $Ln[\log(n+1)]$ and satisfies

$$P^2_{Ln[\log(n+1)]}(x) \asymp \exp(-|x|), \quad |x| \leq Bn, \tag{15}$$

see [7].

Hence we get

$$\frac{\lambda_n(w_\alpha, x)}{\exp(-|x|)} \geq k \cdot n^{1+\frac{\alpha}{2}} \cdot \lambda_{cn\log(n+1)}\left(v^{(\alpha/2)}, \frac{x}{Bn}\right), \quad |x| \leq Bn, \tag{16}$$

further

$$\frac{\lambda n(w_\alpha,x)}{\exp(-|x|)} \geq \frac{\lambda_{kn}(w_\alpha,x)}{\exp(-|x|)} \geq kn^{1+\frac{\alpha}{2}} \cdot \lambda_{cn\log(n+1)}\left(v^{(\alpha/2)}, \frac{x}{Bkn}\right),$$
$$|x| \leq Bkn,$$

where k is arbitrary fixed positive integer. Consequently Lemma 2 (b) is valid for $|x| \leq cn$, where c is an arbitrary large constant.

c) Case $0 < m < 1$. In this case we have

$$\lambda_n(w_\alpha,x) \geq \min_{\substack{T_{n-1}\in\Pi_{n-1}\\ T_{n-1}(x)=1}} \int\limits_{-Bn^{1/m}}^{Bn^{1/m}} |T_{n-1}(t)|^2 w_\alpha(t)dt$$

$$= k \min_{\substack{T_{n-1}\in\Pi_{n-1}\\ T_{n-1}(x)=1}} \int\limits_{-B_1n}^{B_1n} |T_{n-1}(|s|^{1/m})|^2 \cdot |s|^{\frac{\alpha}{2m}+\frac{1}{m}-1} \cdot e^{-|s|}ds$$

$$\geq k \min_{\substack{T_{n/m}\in\Pi_{\frac{n}{m}}\\ T_{n/m}(|x|^m)=1}} \int\limits_{-B_1n}^{B_1n} |T_{n/m}(s)|^2 \cdot |s|^{\frac{\alpha}{2m}+\frac{1}{m}-1} \cdot e^{-|s|}ds$$

$$= k\lambda_{n/m}\left(w_{\frac{\alpha}{m}+\frac{2}{m}-2}, |x|^m\right).$$

Here we can use the result of Lemma 2, (b) and thus we get Lemma 2, (c).

Proof of the sharpness of Nikolskiĭ-type inequality.

Case (i): $p \leq q$.

We will obtain the polynomials R_n^* from D. S. Lubinsky's function G defined by (7). According to [4] Theorem 6 $G(x) \asymp \exp(|x|^m)$, $x \in \mathbb{R}$, and thus if we define R_n^* to be the $n/2$-th partial sum of the power series in (17) then

$$0 < R_n^*(x) \leq K\exp(|x|^m), \quad x \in \mathbb{R}. \tag{17}$$

Moreover, a close inspection of the remainder term $G - R_n^*$ shows that there exists $\varepsilon > 0$ such that $G(x) \leq R_n^*(x) + o(1)$ uniformly for $|x| \leq \varepsilon n^{1/m}$ and $n = 1, 2, \dots$ where $\lim\limits_{n\to\infty} o(1) = 0$. Hence we have

$$\exp(|x|^m) \leq K \cdot R_n^*(x), \quad |x| \leq \varepsilon n^{\frac{1}{m}}. \tag{18}$$

Let $r > 0$. Then we obtain

$$\|R_n^* w_\alpha\|_r \asymp \left\|R_n^* w_\alpha \chi_{[-cn^{1/m},cn^{1/m}]}\right\|_r \asymp \left\|\chi_{[-cn^{1/m},cn^{1/m}]} \cdot v_\alpha\right\|_r \asymp n^{\frac{\alpha}{2m}+\frac{1}{mr}},$$

where $v_\alpha(x) = |x|^{\alpha/2}$, $n = 1, 2, \ldots$ which proves (3) in case $p \le q$.

Case (ii): $p > q$ and $m > 1$. Pick $r_0 > 0$. We will prove (3) by constructing a sequence of polynomials $\{R_n^*\}_{n=1}^\infty$, $\deg R_n^* \le n$, such that for every fixed $r > r_0$

$$\|R_n^* \cdot w_\alpha\|_r \asymp n^{((1/m)-1)r}, \quad n = 1, 2, \ldots \, . \tag{19}$$

Given $r_0 > 0$, let us choose an integer $M > 0$ such that $r_0 > M^{-1}$. Define the weight function u as follows

$$u(x) = |x|^{\frac{\alpha}{2M}} \cdot \exp(-|x|^m/M), \quad x \in \mathbb{R},$$

and let $\{p_n(u, x)\}_{n=0}^\infty$ be the system of polynomials which is orthonormal with respect to u. Define the function $K_n(u)$ by the formula

$$K_n(u, x, y) = \sum_{k=0}^{n-1} p_k(u, x) \cdot p_k(u, y).$$

and let $N = \max\left\{1, \left[\frac{n}{2M}\right]\right\}$. Then

$$R_n^* := \left(\frac{K_N(u, x, x_0)}{K_N(u, x_0, x_0)}\right)^{2M} \tag{20}$$

is a polynomial of degree at most n, where $x_0 \neq 0$ is arbitrary fixed, independent of n. We may assume that $x_0 > 0$.

In what follows we will show hat $\{R_n^*\}_{n=1}^\infty$ satisfies (19). Using orthogonality it follows

$$\|R_n^* w_\alpha\|_{1/M} = \left\{\int_R \frac{K_N^2(u, x, x_0)}{K_N^2(u, x_0, x_0)} u(x) dx\right\}^M = K_N^{-M}(u, x_0, x_0).$$

According to the Lemma 2, (a) we obtain $K_N^{-1}(u, x_0, x_0) = O(N^{(1/m)-1})$. Therefore we obtain

$$\|R_n^* w_\alpha\|_{1/M} \le K n^{((1/m)-1)M}$$

and by (2)

$$\|R_n^* w_\alpha\|_r \le K n^{\left(1-\frac{1}{m}\right)\left(M-\frac{1}{r}\right)} \cdot \|R_n^* w_\alpha\|_{1/M} \le K \cdot n^{((1/m)-1)r} \tag{21}$$

holds for $n = 1, 2, \ldots$. The next step is to show that there exists $0 < \varepsilon < 1$ such that

$$R_n^*(x) \ge \frac{1}{2}, \quad |x - x_0| \le \varepsilon n^{(1/m)-1}. \tag{22}$$

By (20) $(R_n^*)^{\frac{1}{2M}}$ is a polynominal of degree at most n such that $(R_n^*(x_0))^{\frac{1}{2M}} = 1$. Hence for $|x - x_0| \le \varepsilon n^{(1/m)-1}$, $n \ge n_0 = n_0(x_0, m)$,

$$\left|1 - (R_n^*(x))^{\frac{1}{2M}}\right| = \left|\int_x^{x_0} \left[(R_n^*(t))^{1/2M}\right]' dt\right|$$

$$\le \sqrt{|x - x_0|} \cdot \left\{\int_x^{x_0} \left|\left[(R_n^*(t))^{\frac{1}{2M}}\right]'\right|^2 dt\right\}^{\frac{1}{2}}$$

$$\le k\sqrt{|x - x_0|} \left\{\int_x^{x_0} \left|\left[(R_n^*(t))^{\frac{1}{2M}}\right]' \cdot \sqrt{u(t)}\right|^2 dt\right\}^{\frac{1}{2}}$$

$$\le k\sqrt{|x - x_0|} \cdot \left\|\left[(R_n^*(t))^{\frac{1}{2M}}\right]' \cdot \sqrt{u}\right\|_2 .$$

Here, using a Markov–Bernstein type inequality [1], the right-hand side can be estimated in terms of

$$Kn^{(1-(1/m))}\sqrt{|x - x_0|} \cdot \left\|(R_n^*)^{\frac{1}{2M}}\sqrt{u}\right\|_2$$

$$= Kn^{(1-(1/m))} \cdot \sqrt{|x - x_0|} \cdot K_N^{-\frac{1}{2}}(u, x_0, x_0) = K\sqrt{|x - x_0|} \cdot n^{\frac{1-(1/m)}{2}} .$$

Hence we get

$$\left|1 - (R_n^*(x))^{\frac{1}{2M}}\right| = Kn^{\frac{1-(1/m)}{2}} \cdot \sqrt{|x - x_0|} \,,$$

where $|x - x_0| \le \varepsilon n^{(1/m)-1}$; from this (22) immediately follows. From (22) we obtain

$$Kn^{((1/m)-1)r} \le \|R_n^* w_\alpha\|_r, \quad n = 1, 2, \ldots \ . \tag{23}$$

From (21) and (23) the desired (19) follows.

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I. JOÓ
DEPARTMENT OF MATHEMATICS
L. EÖTVÖS UNIVERSITY
1088 BUDAPEST

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