Remark on the paper Á. Száz: Krabbe's generalized functions as convolution quotients

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Let $\mathscr{I}=\langle\alpha,\beta\rangle$ be an arbitrary bounded or unbounded interval of the real line such that $0\in\mathscr{I}$. Let $\mathscr{L}(\mathscr{I})$ be the convolution algebra of locally Lebesgue-integrable complex-valued functions on $\mathscr{I}$. Thus the multiplication in $\mathscr{L}(\mathscr{I})$ is the convolution

$$(fg)(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau \qquad (f,g\in\mathscr{L}(\mathscr{I});\ t\in\mathscr{I}),$$

while the addition and the multiplication by scalars are usually defined. Since $\mathscr{L}(\mathscr{I})$ contains elements which are not divisors of zero, $\mathscr{L}(\mathscr{I})$ has a quotient ring $Q(\mathscr{I})$.

Let $\mathscr{T}(\mathscr{I})$ be the set of functions $\varphi\in\mathscr{L}(\mathscr{I})$ which are infinitely differentiable on $\mathscr{I}$ and such that $\varphi^{(k)}(0)=0$ for every integer $k\geqq 0$.

A mapping $F:\mathscr{T}(\mathscr{I})\to\mathscr{T}(\mathscr{I})$ is called a multiplier operator of $\mathscr{T}(\mathscr{I})$ (see [1]) if

$$F(\varphi\psi)=F(\varphi)\psi$$

holds for all $\varphi,\psi\in\mathscr{T}(\mathscr{I})$. The set of all multiplier operators of $\mathscr{T}(\mathscr{I})$ will be denoted by $\mathscr{G}(\mathscr{I})$.

Á. SZÁSZ has shown ([2]) (after the embedding of $\mathscr{G}(\mathscr{I})$ in $Q(\mathscr{I})$) that

$$\{\mathscr{L}(\mathscr{I}),s\}\subseteqq\mathscr{G}(\mathscr{I}) \tag{1}$$

where $\{\mathscr{L}(\mathscr{I}),s\}$ is the subring of $Q(\mathscr{I})$ generated by $\mathscr{L}(\mathscr{I})$ and $s=\dfrac{1}{\{1\}}$. He has the following question stated: *Is the equality*

$$\mathscr{G}(\mathscr{I})=\{\mathscr{L}(\mathscr{I}),s\}$$

true? In the present note we give a negative answer for this question.

Obviously, it is sufficient to verify only in a special case the relation

$$\{\mathscr{L}(\mathscr{I}),s\}\subset\mathscr{G}(\mathscr{I}). \tag{2}$$

Next we shall suppose that $\mathscr{I}=[0,\infty)$ and in this case $Q(\mathscr{I})$ is the field of Mikusiński operators ([3]).

It should be remarked that for every element a of $\{\mathscr{L}(\mathscr{I}), s\}$ there exists a positive integer k and a continuous function f such that

$$a=s^k f \quad (f\in\mathscr{C}). \tag{3}$$

It is clear that the operator

$$b=\sum_{\nu=0}^{\infty} s^{\nu} e^{-\nu s} \tag{4}$$

cannot be written in the form (3). Consequently $b\notin\{\mathscr{L}(\mathscr{I}), s\}$. However $b\in\mathscr{G}(\mathscr{I})$. Let, namely, φ be an arbitrary element of $\mathscr{T}(\mathscr{I})$. We extend φ for $(-\infty, 0)$ as follows: $\varphi(t)=0$ if $t<0$. Then we obtain

$$b\varphi=\sum_{\nu=0}^{\infty} s^{\nu} e^{-\nu s}\varphi=\left\{\sum_{\nu=0}^{\infty}\varphi^{(\nu)}(t-\nu)\right\}\in\mathscr{T}(\mathscr{I}).$$

Thus $b\in\mathscr{G}(\mathscr{I})$, since, by Theorem 5. of [2], b is an element of $\mathscr{G}(\mathscr{I})$ iff $b\varphi\in\mathscr{T}(\mathscr{I})$ for all $\varphi\in\mathscr{T}(\mathscr{I})$.

References

[1] L. Máté, Multiplier operators and quotient algebra, *Bull. Acad. Polonaise,* **8**, (1965), 523—526.
[2] Á. Szász, Kabbe's generalized functions as convolution quotients. *In this journal*
[3] J. Mikusiński, Operational calculus, *New York,* 1959.

(Received November 8, 1971.)