On the homothetic transformations in areal spaces of the submetric class

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1. Introduction

The homothetic transformations were first studied by E. B. SHANKS [1][1]) in Riemannian spaces. YANO [2] studied the homothetic transformations and group of homothetic transformations more systematically in a Riemannian space and formulated his theory specially by making the use of Lie derivatives. Lateron, these transformations were considered by TAKANO [3] in Finsler spaces and subsequently, the theory of r parameter group G_r of homothetic transformations was given by HIRAMATU [4, 5] in Finsler manifold.

On the other hand, the geometry of an areal space of the submetric class has been studied by KAWAGUCHI and TANDAI [7, 8], GAMA [9, 10, 11, 12], KIKUCHI [13, 14] and many others [17]. IGARASHI [15] has also developed the theory of Lie derivatives in areal spaces.

The prime purpose of the present paper is to discuss the theory of homothetic transformations in areal spaces of the submetric class and to show some of the interesting properties of this transformation in areal spaces of the submetric class which are enjoyed by these spaces in the wide sense because the spaces Riemannian and Finsler are the special cases of an areal space of the submetric class and consequently, the corresponding theory of homothetic transformations in Riemannian and Finsler spaces may be derived from the theory of homothetic transformations in areal spaces of the submetric class.

In what follows, we shall use the same notations and symbolism as those employed by KAWAGUCHI [17] and by previous authors in their studies of areal spaces, without explanations.

2. Some Preliminary results

Let us consider an n-dimensional manifold referred to a coordinate system (x^i) and attach the m-plane direction p^i_α with every point of the manifold. Thus, a manifold, each point of which is associated with a set [2]) (x^i, p^i_α) constituting the so called m-plane element of support of the manifold, is named as an areal space.

[1]) Numbers in brackets refer to the references at the end of the paper.

[2]) The Latin indices $h, i, j, k, \ldots$ run from 1 to N and Greek indices $\alpha, \beta, \gamma, \delta, \ldots$ from 1 to M, throughout this paper.

The fundamental function of this space is denoted by $F(x, p)$, $p \equiv p^i_\alpha$, which is a positive analytical and homogeneous function of degree one in p's.

In such an areal space which is always regular, if there exists an intrinsic symmetric tensor with two indices g_{ij} (or g^{ij}) and whose components, depending on an m-dimensional plane element P_m, can be deduced from the fundamental function $F(x, p)$, the transversal m-vector $G_{i[m]}$, and the metric m-tensor $g_{i[m];j[m]}$ by an algebraic operation, the space is said to be of the submetric class.

In an areal space $A_n^{(m)}$ of the submetric class, the absolute covariant differential of a contravariant vector X^i has been defined as

$$DX^i = X^i_{|k}\,dx^k + X^i|^\lambda_r Dp^r_\lambda, \quad Dp^r_\lambda = \gamma^r_l (dp^l_\lambda + \dots + p^h_\lambda \Gamma^{*l}_{hk}\,dx^k),$$

$$\gamma^r_l = \delta^r_l - \beta^r_l, \quad \beta^r_l = p^r_\alpha p^\alpha_l, \qquad x^k),$$

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$$X^i_{|k} = X^i_{,k} - X^i{}_{;r}^{\lambda} \Gamma^{*r}_{\lambda k} + X^j \Gamma^{*i}_{jk}. \tag{2.1}$$

$$X^i|^\lambda_r = X^i{}_{;r}^{\lambda} + X^j C^i_{j,r}{}^{\lambda}, \tag{2.2}$$

and the symbols the small vertical bar (|) and a long solidus denote the covariant derivatives of X^i with respect to x^k and p^r_λ respectively. In above expressions, the comma and semi-colon indicate the ordinary partial derivatives with respect to x^k and p^r_λ respectively.

Now, in an $A_n^{(m)}$, let us introduce an infinitesimal point transformation.

$$\bar{x}^i = x^i + \xi^i(x)\,dt, \tag{2.3}$$

where dt is an infinitesimal constant and ξ^i is a contravariant vector field defined over the demain $\mathscr{R}$ of the space under consideration, which is independent of the direction and is of atleast class C^2.

With respect to the infinitesimal point transformation (2. 3), IGARASHI [15] has determined the Lie derivatives in an areal space. The Lie derivatives of the connection parameters Γ^{*i}_{jk} of the space $A_n^{(m)}$ are defined by

$$\pounds \Gamma^{*}{}^i_{jk} = \xi^i_{|j|k} + R^i_{jkl}\xi^l + \Gamma^{*}{}^i_{jk;l}{}^{\alpha} \xi^l_{|h} p^h_\alpha, \tag{2.4}$$

and the Lie derivative of a general tensor T^i_j may be defined as

$$\pounds T^i_j = T^i_{j|k}\xi^k + T^i_{j;k}{}^{\alpha} \xi^k_{|h} p^h_\alpha - T^k_j \xi^i_{|k} + T^i_k \xi^k_{|j}, \tag{2.5}$$

where $\pounds$ denotes the operator of the Lie differention and R^i_{jkl} is the curvature tensor of the space $A_n^{(m)}$, defined by Gama [9].

Also, the Lie derivative [16] of the normalized metric tensor g_{ij} [7] is given by

$$\pounds g_{ij} = \xi_{i|j} + \xi_{j|i} + g_{ij;k}{}^{\alpha} \xi^k_{|h} p^h_\alpha. \tag{2.6}$$

On making the use of the formula (2. 1), (2. 5) and as a result of (2. 4), we have the following useful relation:

$$\pounds(g_{ij|k}) - (\pounds g_{ij})_{|k} = (-\delta^h_i g_{lj} - \delta^h_j g_{il} - g_{ij;l}{}^{\alpha} p^h_\alpha)(\pounds \Gamma^{*}{}^l_{hk}), \tag{2.7}$$

which may also be directly obtained as an immediate consequence of the Lemma 3 of IGARASHI [15].

On the other hand the application of two operators $\pounds$ and $|$ on a contravariant vector X^i, after a long calculation, gives the commutation formula:

$$\pounds(X^i|_r^\lambda)-(\pounds X^i)|_r^\lambda = X^j \pounds C_{j,}^i{}_r^\lambda, \tag{2.8}$$

where we have the use of relations (2. 2), (2. 5) and the indentity $\pounds(X^i;_\nu^\lambda) = (\pounds X^i);_r^\lambda$.

Furthermore, we define an operator $]\!]$ for a generalized geometric object Ω by

$$\Omega]\!]_i^\alpha = F(\Omega;_i^\alpha),$$

so that, it is interesting to note that if we apply the commutator of two operators $\pounds$ and $]\!]$ on a general tensor T_j^i, then after some simple calculation, we have

$$\pounds(T_j^i]\!]_k^\alpha)-(\pounds T_j^i)]\!]_k^\alpha = (T_j^i]\!]_k^\alpha)\beta_r^h \xi^r{}_{|h}, \tag{2.9}$$

where we have used the definition of $]\!]$ and the identity

$$\pounds(X^i;_r^\lambda) = (\pounds X^i);_r^\lambda.$$

We shall now define the homothetic transformations in areal spaces of the submetric class.

3. Definition of homothetic transformations

Two distinct n-dimensional areal spaces $A_n^{(m)}$ and $\bar{A}_n^{(m)}$ of the submetric class with the same system of coordinates are said to be conformally related [12, 13], if their respective normalized metric tensor g_{ij} and $\bar{g}_{ij}$ are connected by the relation

$$\bar{g}_{ij} = e^{2\sigma} g_{ij}, \tag{3.1}$$

where $e^{2\sigma}$ is a factor of proportionality and σ is atmost a point function. In above relation, if we consider the function σ a constant instead of taking it as a scalar point function, the transformation (3. 1) becomes homothetic one in the sense of Shanks.

In relation (3. 1), if we assume the constant σ equal to zero, then clearly, $\bar{g}_{ij}= =g_{ij}$ and the respective connection parameters $\bar{\Gamma}^*{}_{ij}^h$ and $\Gamma^*{}_{ij}^h$ of the spaces $\bar{A}_n^{(m)}$ and $A_n^{(m)}$ also become equal, i.e. $\bar{\Gamma}^*{}_{ij}^h=\Gamma^*{}_{ij}^h$, consequently, we have $\pounds g_{ij}=0$ and $\pounds\Gamma^*{}_{ij}^h=0$. Hence in the case $\sigma=0$, the transformation (3. 1) becomes an areal motion [15]. Therefore, throughout this paper, we shall consider such a transformation (3. 1), in which σ is a non-zero constant.

Now for the sake of conveniency, we write the relation (3. 1) into the form

$$\bar{g}_{ij} = 2cg_{ij}, \tag{3.2}$$

where c is a non-zero constant, because of the reason that in the case $c=0$, our transformation (3. 2) becomes an areal motion in the space $A_n^{(m)}$. Thus, for $c\neq 0$, the transformation (3. 2) will be known as a proper homothetic transformation in the space $A_n^{(m)}$ and c will be called a homothetic constant.

4. Homothetic transformations between the areal spaces of the submetric class

Let us have the Euler vector E_i, defined by

$$E_i = F^{-1}(F;_i^\alpha,_\alpha - F,_i). \tag{4. 1}$$

It is also well known that under the homothetic transformation (3. 1), the fundamental functions $F(x, p)$ and $\bar{F}(x, p)$ of the spaces $A_n^{(m)}$ and $\bar{A}_n^{(m)}$ respectively are related by

$$\bar{F} = KF, \quad K = e^{m\sigma}. \tag{4. 2}$$

On differentiating this, we find that

$$\bar{F},_i = K,_i F + KF,_i, \quad \bar{F};_i^\alpha,_\alpha = K,_j p_\alpha^j F;_i^\alpha + KF;_i^\alpha,_\alpha,$$

where $K,_i = \partial K/\partial x^i$ and $F;_i^\alpha,_\alpha = \partial F;_i^\alpha/\partial u^\alpha$. From these expressions, we may deduce the relation

$$\bar{F};_i^\alpha,_\alpha - \bar{F},_i = K(F;_i^\alpha,_\alpha - F,_i) + (K,_j p_\alpha^j F;_i^\alpha - K,_i F). \tag{4. 3}$$

But for a non-zero constant σ, K is also a constant, so the terms under bracket on the R.H.S. of the relation (4. 3) vanish, consequently, we obtain

$$\bar{F};_i^\alpha,_\alpha - \bar{F}_i = K(F;_i^\alpha,_\alpha - F,_i).$$

On substituting the desired expression for K from (4. 2) in this relation and by help of (4. 1), we get finally

$$\bar{E}_i = E_i, \tag{4. 4}$$

which clearly shows that under the proper homothetic transformation (3. 1), the Eulerian vectors are invariant. Conversely, if the Euler vectors are invariant, the expression under bracket on the R.H.S. of (4. 3) vanishes, i.e., $K,_j p_\alpha^j F;_i^\alpha - K,_i F = 0$. This gives

$$K,_i(p_\alpha^i F;_j^\alpha - F) = 0.$$

From which we conclude that K is a constant, say $K,_i = 0$, because $p_\alpha^i F;_j^\alpha - F \neq 0$. Hence, the transformation is a proper homothetic one. Thus, we have the

Theorem 4. 1. *If two areal spaces $A_n^{(m)}$ and $\bar{A}_n^{(m)}$ of the submetric class admit a proper homothetic transformation, the Eulerian vectors are invariant under this transformation, and the converse is also true.*

Moreover, the vanishing of Euler vector (4. 1) in an $A_n^{(m)}$ characterizes the extremal subspaces. Therefore, from (4. 4), the vanishing of either Euler vector of the spaces under consideration reduces the other to zero.

Theorem 4. 2. *When two areal spaces $A_n^{(m)}$ and $\bar{A}_n^{(m)}$ admit the transformation* (3. 1), *the transformation leaves the extremal subspaces invariant, if and only if the transformation* (3. 1) *is a proper homothetic one.*

It is well known that an areal space $A_n^{(m)}$ $(n>2)$ of the submetric class of constant curvature is characterized by the relation

$$R^i_{jkh} = R(x, p)(\delta^i_h g_{jk} - \delta^i_k g_{jh}), \qquad R(x, p) \neq 0, \tag{4. 5}$$

where $R(x, p)$ is the Riemannian curvature of the space $A_n^{(m)}$ at the point x^i.

On the other hand, KIKUCHI [14] has shown that if the relation (4. 5) holds good, the space $A_n^{(m)}$ is a Riemannian space with constant curvature. Therefore, in analogy with the theorem 3 of SHANKS [1]. The following theorem may be established without difficulty:

Theorem 4. 3. *Two areal spaces $A_n^{(m)}$ and $\bar{A}_n^{(m)}$ of the submetric class of the same non-zero constant curvature admit no proper homothetic transformation between them, while two spaces $A_n^{(m)}$ and $\bar{A}_n^{(m)}$ with unequal positive (or negative) constant curvature admit a proper homothetic transformation. The constant exponent σ is uniquely determined by the relation*

$$\sigma = \frac{1}{2} \log \frac{R(x,p)}{\bar{R}(x,p)},$$

where $R(x,p)$ and $\bar{R}(x,p)$ are the Riemannian curvatures of the spaces $A_n^{(m)}$ and $\bar{A}_n^{(m)}$, respectively.

5. The homothetic transformations of an areal space of the submetric class with itself

In this section, we discuss the homothetic transformations to which a space $A_n^{(m)}$ admits into itself. For this purpose, if we make the use of Lie derivatives, then the proper homothetic transformation relation (3. 2) may be characterized by

$$\text{(5. 1)} \qquad \pounds g_{ij} = 2c g_{ij},$$

and the transformation is now called a proper infinitesimal homothetic transformation with a homothetic constant c.

Employing the relation (5. 1) in (2. 7) and taking note of the fact that for covariant differential in $A_n^{(m)}$, the normalized metric tensor g_{ij} behaves as a constant, we see that the L.H.S. reduces to zero, so we get

$$\text{(5. 2)} \qquad (-\delta_i^h g_{lj} - \delta_j^h g_{il} - g_{ij};{}_l^\alpha p_\alpha^h)(\pounds \Gamma^{*l}_{hk}) = 0,$$

from (5. 2), if we assume $(-\delta_i^h g_{lj} - \delta_j^h g_{il} - g_{ij};{}_l^\alpha p_\alpha^h) \neq 0$, which is quite obvious, we get $\pounds \Gamma^{*l}_{hk} = 0$. Hence we have the

Theorem 5. 1. *When an $A_n^{(m)}$ admits a proper infinitesimal homothetic transformation, it is necessary and sufficient that the transformation be an areal motion at the same time, provided that $\|(-\delta_i^h g_{lj} - \delta_j^h g_{il} - g_{ij};{}_l^\alpha p_\alpha^h)\| \neq 0$.*

Further, it is no more difficult that after some simple calculation, the relation (2. 6) can be written as

$$\text{(5. 3)} \qquad \pounds g_{ij} = g_{ij},_k \xi^k + g_{kj} \xi^k,_i + g_{ik} \xi^k,_j + g_{ij};{}_h^\alpha \xi,{}_k^h p_\alpha^k.$$

Introducing (5. 1) in (5. 3), we obtain

$$\text{(5. 4)} \qquad 2c g_{ij} = g_{ij},_k \xi^k + g_{kj} \xi^k,_i + g_{ik} \xi^k,_j + g_{ij};{}_h^\alpha \xi^h,_k p_\alpha^k.$$

Now, we consider a one parameter group of transformation and choose such a coordinate system with respect to which $\xi^i = \delta^i_1$, then the infinitesimal transformations (2. 3), which are the finite equations of the group, become

$$\bar{x}^i = x^i + \delta^i_1\, dt. \tag{5. 5}$$

In such a case from (5. 4), we find that

$$g_{ij},_1 = 2cg_{ij}. \tag{5. 6}$$

Consequently, from (5. 6), we have

$$g_{ij}(x^i, p^i_\alpha) = e^{2cx^1}\mathfrak{G}_{ij}(x^2, x^3, \ldots, x^n, p^i_\alpha), \tag{5. 7}$$

where $\mathfrak{G}_{ij}$ is a homogeneous function of degree zero in the p's. Conversely, if there exists a coordinate system in which the normalized metric tensor g_{ij}, which serves as a fundamental metric tensor in an areal space $A_n^{(m)}$ of the submetric class, takes the form (5. 7), then the space $A_n^{(m)}$ admits a one parameter group of proper homothetic transformations generated by (5. 5). Thus, we have the

Theorem 5. 2. *If an areal space $A_n^{(m)}$ of the submetric class admits an infinitesimal proper homothetic transformation, then $A_n^{(m)}$ admits also a one parameter group of proper homothetic transformations generated by the infinitesimal one.*

Theorem 5. 3. *When an areal space $A_n^{(m)}$ of the submetric class admits a one parameter group of proper infinitesimal homothetic transformation, it is necessary and sufficient that there exists a coordinate system with respect to which the normalized metric tensor of $A_n^{(m)}$ takes the form* (5. 7).

Furthermore, applying $\xi^i = \delta^i_1$ in (2. 4), we get

$$\pounds\Gamma^{*}{}^{i}_{jk} = R^i_{jk1}. \tag{5. 8}$$

However, we notice that if an $A_n^{(m)}$ admits a proper infinitesimal homothetic transformation, $\pounds\Gamma^{*}{}^{i}_{jk}=0$, by reason of this fact, from (5. 8), we see that $R^i_{jk1}=0$, but $R^i_{jkl} = -R^i_{jlk}$. Therefore $R^i_{jk1} = -R^i_{j1k} = 0$.

We now remember that the curvature tensor R^i_{jkl} satisfies the identity

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0.$$

If we put $l=1$ in this identity, then under the above facts, we finally have

$$R^i_{k1j} = R^i_{1kj} = R^i_{jk1} = 0.$$

Hence, we have the

Theorem 5. 4. *If an areal space $A_n^{(m)}$ of the submetric class admits a one parameter group of proper infinitesimal homothetic transformation, it is necessary and sufficient that those components of the curvature tensor R^i_{jkl} of $A_n^{(m)}$, which are necessarily the function of x^1 in their lower indices, vanish identically.*

In an areal space $A_n^{(m)}$ of the submetric class, the subspaces defined by

$$H^i_{\alpha\beta} = \gamma^i_j(p^j_{\alpha\beta} + \Gamma^{*}{}^{j}_{hk}\, p^h_\alpha p^k_\beta) = 0, \quad p^j_{\alpha\beta} = \frac{\partial x^j}{\partial u^\alpha\, \partial u^\beta},$$

are said to be totally geodesic.

Using the formula (2. 5), we have

$$\pounds \gamma^i_j = \gamma^i_{j|k}\xi^k + \gamma^i_j{;}^{\alpha}_{k}\xi^k_{|h} p^h_\alpha - \gamma^k_j \xi^i_{|k} + \gamma^i_k \xi^k_{|j}.$$

But $\gamma^i_{j|k}=0$, so on putting $\xi^i=\delta^i_1$, we find that $\pounds\gamma^i_j=0$. Under this fact, applying the transformation (2. 3) and putting $\xi^i=\delta^i_1$, if we use the condition $\bar{\Gamma}^{*i}_{jk}=\Gamma^{*i}_{jk}$ (because $\pounds\Gamma^{*i}_{jk}=0$) by theorem 5. 1, then we can see with ease that the totally geodesic subspaces remains invariant. Hence, we can give the

Theorem 5. 5. *If an areal space $A^{(m)}_n$ of the submetric class admits a one parameter group of proper infinitesimal homothetic transformation, the transformation generated by the infinitesimal one leaves the totally geodesic subspaces invariant.*

Next, we take a coordinate system in which $\xi^i=x^i$, then because of the condition $g_{ij}{;}^{\alpha}_{k}p^k_\alpha=0$, relation (5. 4) gives us

$$g_{ij},_k x^k = 2(c-1)g_{ij},$$

which shows that normalized metric tensor g_{ij} is a homogeneous function of degree $2(c-1)$ with respect to ξ^i. Hence, we have the

Theorem 5. 6. *In order that an areal space $A^{(m)}_n$ of the submetric class admits a one parameter group of proper infinitesimal homothetic transformation with a homothetic constant c, it is necessary and sufficient that there exists a coordinate system in which the components of a normalized metric tensor g_{ij} are homogeneous functions of degree $2(c-1)$ of the coordinate variables x^i.*

Remark 1. It is interesting to point out that for the case $m=1$, the areal space $A^{(m)}_n$ becomes the Finsler space in particular. In such a case, theorems 5. 2, 5. 3 and 5. 6 hold good coinciding with the corresponding theorems of TAKANO [3] in Finsler spaces. The Riemannian space is one of the special case of areal space $A^{(m)}_n$ for $C_{ij},^{\alpha}_{k}=0$, therefore, these theorems also hold good for the Riemannian spaces in particular.

Remark 2. In the above discussion, since the proof of the theorems 5. 4 and 5. 5 is based on the condition $\pounds\Gamma^{*i}_{jk}=0$, which is an essential condition for the areal motion as well as for the proper infinitesimal homothetic transformations, therefore these two theorems also hold good for the one parameter group of areal motion in the space $A^{(m)}_n$.

6. Some additional properties of the homothetic transformations

The fundamental function $F(x, p)$ of the space $A^{(m)}_n$ is changed as

$$\bar{F}(x, p)=mcF(x, p)$$

under the transformation (3. 2). If we make the use of Lie-derivatives, then, under the proper homothetic transformations (3. 2), the variation in the function can be written as

$$\pounds F = mcF. \tag{6. 1}$$

In an areal space $A_n^{(m)}$, the area of an infinitesimal domain on an m-dimensional sub-space $x^i = x^i(u^\alpha)$, $\alpha = 1, 2, \ldots, m$, is given by

$$dS = F(x, p)(du)^m, \quad \text{where} \quad (du)^m = du^1 du^2 \ldots du^m, \tag{6. 2}$$

by means of *a priori* given function $F(x, p)$. The present author [16] has also shown that under the infinitesimal point transformation (2. 3), the fundamental function $F(x, p)$ varies as

$$\bar{\bar{F}}(x, p) = F(x, p) + \pounds F(x, p)\, dt,$$

where

$$\pounds F(x, p) = (F;{}_i^\alpha)\xi^i{}_{|j}\, p^j_\alpha. \tag{6. 3}$$

Employing the transformation (2. 3), the area dS on an infinitesimal domain is transformed as

$$\overline{\overline{dS}} = dS + (\pounds F\, dt)\,(du)^m.$$

Introducing (6. 1) and (6. 3) in this relation and on making the use of relation $F;{}_i^\alpha = Fp_i^\alpha$, if we note that $p_i^\alpha p_\alpha^j = \beta_i^j$, we find that

$$\frac{\overline{\overline{dS}}}{dS} = 1 + mc\, dt, \quad \text{or} \quad \frac{\overline{\overline{dS}}}{dS} = 1 + \beta_i^j \xi^i{}_{|j}\, dt,$$

consequently, we obtain that

$$mc = \beta_i^j \xi^i{}_{|j}. \tag{6. 4}$$

From this relation, it is obviously presumptive that for a constant c, the quantity $\beta_i^j \xi^i{}_{|j}$ constituting an invariant should invariably be a constant. Thus, we can state the

Theorem 6. 1. *In order that an areal space $A_n^{(m)}$ of the submetric class admits a proper infinitesimal homothetic transformation, it is necessary and sufficient that the invariant $\beta_i^j \xi^i{}_{|j}$ should necessarily be a constant for a properly chosen element of support p_α^i.*

Furthermore we may also have the

Theorem 6. 2. *If an areal space $A_n^{(m)}$ of the submetric class admits a proper infinitesimal homothetic transformation* (5. 1), *then choosing suitably the plane element of support p_α^i, the homothetic constant c is uniquely determined by the relation.*

$$c = \frac{1}{m} \beta_i^j \xi^i{}_{|j}.$$

Remark 3. In the particular case, when $m = 1$, the areal space $A_n^{(m)}$ of the submetric class becomes a Finsler one, and the above theorem coincide with the corresponding theorem 5 of TAKANO [3] in Finsler spaces.

Now, we apply the commutation formula (2. 9) for the normalized metric tensor g_{ij}, we obtain

$$\pounds(g_{ij} \rrbracket_k^{(\alpha)}) - (\pounds g_{ij}) \rrbracket_k^\alpha = (g_{ij} \rrbracket_k^{(\alpha)})\beta_r^h \xi^r{}_{|h}.$$

In case of the proper infinitesimal homothetic transformation (5. 1), the above relation gives the result

$$\pounds(g_{ij}]\!]^{\alpha}_{k}) = (2+m)cg_{ij}]\!]^{\alpha}_{k}, \tag{6. 5}$$

where we have introduced the relation (6. 4). In the same way, again operating on the same commutators for the tensor $g_{ij}]\!]^{\alpha}_{k}$, and making use of the relations (2. 9), (6. 5), and (6. 4), we shall find that

$$\pounds(g_{ij}]\!]^{\alpha}_{k}]\!]^{\beta}_{l}) = (2+2m)cg_{ij}]\!]^{\alpha}_{k}]\!]^{\beta}_{l}. \tag{6. 6}$$

In any manner, from the definition of homothetic transformations, we have obtained the relations (6. 5), (6. 6). Likewise, the following inductive relations hold good:

$$\begin{cases} \pounds g_{ij} = 2cg_{ij}, \\ \pounds(g_{ij}]\!]^{\alpha}_{k}) = (2+m)cg_{ij}]\!]^{\alpha}_{k}, \\ \pounds(g_{ij}]\!]^{\alpha}_{k}]\!]^{\beta}_{l}) = (2+2m)cg_{ij}]\!]^{\alpha}_{k}]\!]^{\beta}_{l}. \end{cases} \tag{6. 7}$$

Consequently, for an arbitrary number r, it can be no more difficult to put inductively the relation

$$\pounds(g_{ij}]\!]^{\alpha_1}_{k_1}]\!]^{\alpha_2}_{k_2}\cdots]\!]^{\alpha_\gamma}_{k_\gamma}) = (2+\gamma m)cg_{ij}]\!]^{\alpha_1}_{k_1}]\!]^{\alpha_2}_{k_2}\cdots]\!]^{\alpha_\gamma}_{k_\gamma}, \tag{6. 8}$$

where r takes the values 0, 1, 2, ... successively, for the relation (6. 8) can easily be proved as a result of (2. 9). Thus, we have the

Theorem 6. 3. *If an areal space $A_n^{(m)}$ of the submetric class admits a proper infinitesimal homothetic transformation* (5. 1), *a set of relations* (6. 7) *is identically satisfied one after another, and in general the tensors $g_{ij}]\!]^{\alpha_1}_{k_1}]\!]^{\alpha_2}_{k_2}\ldots$, derived successively from the normalized metric tensor g_{ij} by the well defined operator $]\!]$, satisfy the relation* (6. 8) *with respect to the operator $\pounds$ of Lie-derivation.*

Remark 4. In the special case $m=1$, the space $A_n^{(m)}$ under consideration becomes a Finsler space, and then, above theorem coincides with the theorem 6 of TAKANO [3] in Finsler spaces.

On applying the formula (2. 8) for the normalized metric tensor g_{ij}, we can immediately have

$$\pounds(g_{ij}|^{\lambda}_{r}) - (\pounds g_{ij})|^{\lambda}_{r} = -g_{nj}\pounds C^{n}_{i},^{\lambda}_{r} - g_{in}\pounds C^{n}_{j},^{\lambda}_{r}.$$

In our present case, the use of the transformation (5. 1) and the implication of the property $g_{ij}|^{\lambda}_{r} = 0$ in above relation yields the result

$$g_{nj}\pounds C^{n}_{i},^{\lambda}_{r} + g_{in}\pounds C^{n}_{j},^{\lambda}_{r} = 0. \tag{6. 9}$$

Thus, we see that the relation (6. 9) universally holds good in the areal spaces of the submetric class admitting proper infinitesimal homothetic transformations.

Adding $(\pounds g_{nj})C^{n}_{i},^{\lambda}_{r} + (\pounds g_{in})C^{n}_{j},^{\lambda}_{r}$ on both the sides of (6. 9) and making the suitable arrangement after employing the transformation (5. 1), we obtain

$$\pounds C_{ij},^{\lambda}_{r} + \pounds C_{ji},^{\lambda}_{r} = 2c(C_{ij},^{\lambda}_{r} + C_{ji},^{\lambda}_{r}). \tag{6. 10}$$

Now, in the case when an $A_n^{(m)}$ is an areal space of the metric class, the torsion tensor $C_{ij},{}_r^\lambda \equiv g_{nj} C_i^n,{}_r^\lambda$ is symmetric with respect to the indices i and j, by virtue of the theorem of H. IWAMOTO [6]. Therefore, from (6. 10), we get atonce

$$\pounds C_{ij},{}_r^\lambda = 2c C_{ij},{}_r^\lambda. \tag{6. 11}$$

Hence, we can state that when an $A_n^{(m)}$ admits a proper infinitesimal homothetic transformation (5. 1) and if the space is of the metric class, the relation (6. 11) holds good for a homothetic constant c.

Furthermore, we also notice that TANDAI [8] has already shown that the space $A_n^{(m)}$ is always Riemannian one, if $g_{ij};{}_k^\alpha = 0$, i.e., $C_{ij},{}_k^\alpha = 0$. In such a case, from (6. 11), we find that $\pounds C_{ij},{}_k^\alpha = 0$. Hence, we may also have the

Theorem 6. 4. *A necessary and sufficient condition that an areal space $A_n^{(m)}$ of the submetric class becomes a Riemannian space is that $C_{ij},{}_k^\alpha = 0$ holds good in the space, but if the space admits a proper infinitesimal homothetic transformation, this condition may be replaced with $\pounds C_{ij},{}_k^\alpha = 0$, so that the space $A_n^{(m)}$ becomes a Riemannian space admitting proper homothetic transformation.*

7. Further discussion

In this section, we devote ourselves to ensure some more characteristic properties of the homothetic transformations in an $A_n^{(m)}$.

Let us have the identities

$$\xi^h_{|i|j|k} - \xi^h_{|i|k|j} = R^h_{ljk} \xi^l_{|i} - R^l_{ijk} \xi^h_{|l} - (\xi^h_{|i};{}_l^\alpha) R^l_{\alpha jk}, \tag{7. 1}$$

and

$$R^h_{ijk|l} + R^h_{ikl|j} + R^h_{ilj|k} + \Gamma^{*h}_{ij};{}_m^\alpha R^m_{\alpha kl} + \Gamma^{*h}_{ik};{}_m^\alpha R^m_{\alpha lj} + \Gamma^{*h}_{il};{}_m^\alpha R^m_{\alpha jk} = 0 \tag{7. 2}$$

With the help of (7. 1) and (7. 2), after some long calculation, we find that

$$(\pounds \Gamma^{*h}_{ij})_{|k} - (\pounds \Gamma^{*h}_{ik})_{|j} = \pounds R^h_{ijk} + \Gamma^{*h}_{ij};{}_l^\alpha (\pounds \Gamma^{*l}_{rk}) p^r_\alpha - \Gamma^{*h}_{ik};{}_l^\alpha (\pounds \Gamma^{*l}_{rj}) p^r_\alpha, \tag{7. 3}$$

where we have used the relations

$$\xi^h_{|i};{}_l^\alpha = (\Gamma^{*h}_{im};{}_l^\alpha) \xi^m,$$

and

$$R^h_{ijk};{}_l^\alpha = (\Gamma^{*h}_{ij};{}_l^\alpha)_{|k} - (\Gamma^{*h}_{ik};{}_l^\alpha)_{|j} - \Gamma^{*h}_{ij};{}_m^\beta \Gamma^{*m}_{\beta k};{}_l^\alpha + \Gamma^{*h}_{ik};{}_m^\beta \Gamma^{*m}_{\beta j};{}_l^\alpha.$$

From (7. 3), we can see with ease that

$$\pounds R^h_{ijk} = (\pounds \Gamma^{*h}_{ij})_{|k} - (\pounds \Gamma^{*h}_{ik})_{|j} - \Gamma^{*h}_{ij};{}_l^\alpha (\pounds \Gamma^{*l}_{rk}) p^r_\alpha + \Gamma^{*h}_{ik};{}_l^\alpha (\pounds \Gamma^{*l}_{rj}) p^r_\alpha.$$

Consequently, in our present case, when an $A_n^{(m)}$ admits a proper infinitesimal homothetic transformation, above relation gives us

$$\pounds R^h_{ijk} = 0, \tag{7. 4}$$

because of the condition $\pounds \Gamma^{*h}_{ij} = 0$. Using the formula (2. 5) for the tensor R^h_{ijk} and employing the condition (7. 4), if we put $\xi^i = \delta^i_1$ in the result, we get atonce

$$R^h_{ijk|1} = 0. \tag{7. 5}$$

Also, for the tensor R^h_{ijk}, we have the relation

$$R^h_{ijk|m|1} - R^h_{ijk|1|m} = R^l_{ijk} R^h_{lm1} - R^h_{ljk} R^l_{im1} - R^h_{ilk} R^l_{jm1} - R^h_{ijl} R^l_{km1} - R^h_{ijk};^{\alpha}_{l} R^l_{\alpha m1}.$$

By virtue of the theorem 5. 4 and due to the condition (7. 5), above relation yields the result

$$R^h_{ijk|m|1} = 0. \tag{7. 6}$$

Similarly, we can further determine that

$$R^h_{ijk|m|n|1} = 0. \tag{7. 7}$$

Hence, combining the results (7. 5), (7. 6), (7. 7) with the theorem (5. 4), we can enunciate the

Theorem 7. 1. *If an areal space $A^{(m)}_n$ of the submetric class admits a one parameter group of proper infinitesimal homothetic transformation, the following set of relations holds good:*

$$R^h_{ij1} = 0, \quad R^h_{ijk|1} = 0, \quad R^h_{ijk|m|1} = 0, \quad R^h_{ijk|m|n|1} = 0, \ \ldots.$$

In an areal space $A^{(m)}_n$ of the submetric class, we have the curvature tensor K^h_{ijk}, which is defined by

$$K^h_{ijk} = R^h_{ijk} + C^h_i,^{\lambda}_r R^r_{\lambda jk}, \tag{7. 8}$$

where

$$R^r_{\lambda jk} = R^r_{sjk} p^s_{\lambda}.$$

Applying the operator $\pounds$ on both the sides of (7. 8) and employing the condition (7. 5), we obtain

$$\pounds K^h_{ijk} = (\pounds C^h_i,^{\lambda}_r) R^r_{\lambda jk}. \tag{7. 9}$$

On one hand, if we introduce the condition $\pounds C^h_i,^{\lambda}_r = C^h_i,^{\lambda}_r$ into the above relation, we shall have

$$\pounds K^h_{ijk} = C^h_i,^{\lambda}_r R^r_{\lambda jk},$$

or

$$\pounds K^h_{ijk} = K^h_{ijk} - R^h_{ijk}, \tag{7. 10}$$

where we have used the relation (7. 8). Thus, we have the

Theorem 7. 2. *When an areal space $A^{(m)}_n$ of the submetric class admits a proper infinitesimal homothetic transformation, the Lie derivative of the curvature tensor K^h_{ijk} is given by* (7. 10).

On the other hand, if the space $A^{(m)}_n$ is the Riemannian space, $\pounds C^h_i,^{\lambda}_r = 0$ by virtue of the theorem 6. 4, and consequently, from (7. 9), $\pounds K^h_{ijk} = 0$. Hence, the Lie derivative of the curvature tensor K^h_{ijk} vanishes in the case, when an $A^{(m)}_n$ is a Riemannian space admitting a proper infinitesimal homothetic transformation.

Conclusion. In the author's view, it is worthwhile essential to point out some characteristic properties of the homothetic transformations in the space under consideration, as it leaves the general discussion very interesting for the researchers of geometry by stand point of view that for the case of parameters m equal to one, the m-plane element p^i_α reduces to $\dot{x}^i$, and the tensor $C_{ij};^\alpha_k \equiv \frac{1}{2} g_{ij};^\alpha_k$ of $A_n^{(m)}$ becomes $C_{ijk} = \frac{1}{2} \partial g_{ij}/\partial \dot{x}^k$, and $\pounds C_{ij},^\alpha_k = \pounds C_{ijk}$ (for $\alpha = 1$).

Thus, evidently, our space $A_n^{(m)}$ is now a Finsler space in particular. In such a case, it can obviously be seen with ease by putting $m=1$ and making the suitable changes in the results, almost all the theorems of this paper are transformed into the corresponding theorems holding good in Finsler spaces.

Hence, conclusively for $m=1$ in an areal space of the submetric class admitting proper infinitesimal homothetic transformations, the present theory becomes to coincide with the theory of homothetic transformations in Finsler spaces studied by Takano [3]. In the case $\pounds C_{ij},^\alpha_k = 0$, the theory of homothetic transformations in an $A_n^{(m)}$ coincides with the corresponding theory of homothetic transformations in Riemannian spaces studied by YANO [2].

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