Logarithmic mean function of entire functions defined by Dirichlet series

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1. Let E be the set of mapping $f: C \to C$ (C is the complex field) such that the image under f of an element $s \in C$ is
$f(s) = \sum_{n \in N} a_n e^{s\lambda_n}$ with

$$\limsup_{n \to +\infty} \frac{\log n}{\lambda_n} = D < +\infty, \tag{1.1}$$

and $\sigma_c^f = +\infty$ (σ_c^f is the abscissa of convergence of the Dirichlet series defining f); N is the set of natural numbers $0, 1, 2, \ldots, \langle \lambda_n : n \in N \rangle$ is a strictly increasing unbounded sequence of nonnegative reals, $s = \sigma + it$, $\sigma, t \in R$ (R is the field of reals), and $\langle a_n : n \in N \rangle$ is a sequence in C. Since the Dirichlet series defining f converges for each complex s, f is an entire function. Also, since $D < +\infty$, we have ([1], p. 168) $\sigma_a^f = +\infty$ (σ_a^f is the abscissa of absolute convergence of the Dirichlet series defining f), and that f is bounded on each vertical line $\sigma = \sigma_0$.

In 1914, HARDY ([2], 270) defined the mean value of the modulus of an analytic function and studied some of its properties. This lead to various persons to study the mean values of entire functions and their derivatives. Although all sorts of means of entire functions were studied but the authors are not aware if any body has ever studied the logarithmic mean of entire functions defined by Dirichlet series. *We, therefore, in this paper define the logarithmic mean function of an entire function $f \in E$ and study some of its properties.*

Definition 1: For any $f \in E$, we define its logarithmic mean function L as

$$L(\sigma, f) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} \log |f(\sigma + it)|\, dt, \qquad \forall \sigma < \sigma_c. \tag{1.2}$$

Following theorems throw some light on the properties of the logarithmic mean function L of entire functions $f \in E$.

Theorem 1. *If L is the logarithmic mean function of an entire function $f \in E$, then L is a steadily increasing function and* $\log L$ *is a convex function of σ.*

PROOF. In order to prove this theorem we shall follow the method of TITCHMARSH ([3], p. 174). Let $\sigma_1, \sigma_2, \sigma_3 \in R$ be such that $0 < \sigma_1 < \sigma_2 < \sigma_3$. Also let $\Phi: R \to C$

and $\Psi: C \to C$ be two functions defined, respectively, as follows:

$$\Phi(t_2) = \frac{\log |f(\sigma_2 + it_2)|}{f(\sigma_2 + it_2)}, \qquad \forall t_2 \in R,$$

$$\Psi(s) = \lim_{T \to +\infty} \frac{1}{2T} \int_{-T}^{T} f(s + it_2) \Phi(t_2)\, dt_2, \qquad \forall s \in C.$$

It is clear from the definition of Ψ that it is analytic in the half plane $\mathrm{Re}\,(s) \leqq \sigma_3$, and that $|\Psi|$ attains its supremum on the boundary $\mathrm{Re}\,(s) = \sigma_3$, say at $s = \sigma_3 + it_3$. Hence

$$L(\sigma_2, f) = \Psi(\sigma_2) \leqq |\Psi(\sigma_3 + it_3)| \leqq L(\sigma_3, f),$$

which shows that L increases steadily with σ.

We now choose α so that $e^{\alpha\sigma_1} L(\sigma_1, f) = e^{\alpha\sigma_3} L(\sigma_3, f)$. Then

$$e^{\alpha\sigma_2} L(\sigma_2, f) = e^{\alpha\sigma_2} \Psi(\sigma_2) \leqq \sup_{\sigma_1 \leqq \mathrm{Re}(\varrho) \leqq \sigma_3} |e^{\alpha s} \Psi(\sigma_1)| \leqq e^{\alpha\sigma_1} |\Psi(\sigma_1)| \leqq e^{\alpha\sigma_1} L(\sigma_1, f),$$

whence

$$e^{\alpha\sigma_2} L(\sigma_2, f) \leqq e^{\alpha\sigma_1} L(\sigma_1, f),$$

which gives,

$$\log \left(\frac{L(\sigma_2, f)}{L(\sigma_1, f)} \right) \leqq \alpha(\sigma_1 - \sigma_2). \tag{1. 3}$$

Since, by definition, $\alpha = \dfrac{1}{\sigma_1 - \sigma_3} \log \left(\dfrac{L(\sigma_3, f)}{L(\sigma_1, f)} \right)$, it follows, from (1. 3), that

$$\log \left(\frac{L(\sigma_2, f)}{L(\sigma_1, f)} \right) \leqq \frac{\sigma_1 - \sigma_2}{\sigma_1 - \sigma_3} \log \left(\frac{L(\sigma_3, f)}{L(\sigma_1, f)} \right),$$

or

$$\log L(\sigma_2, f) \leqq \frac{\sigma_3 - \sigma_2}{\sigma_3 - \sigma_1} \log L(\sigma_1, f) + \frac{\sigma_2 - \sigma_1}{\sigma_3 - \sigma_1} \log L(\sigma_3, f),$$

which proves the convexity of $\log L$.

Theorem 2. *If L is the logarithmic mean function of an entire function $f \in E$ of Ritt order ϱ and lower order λ, M is the supremum function of $|f|$ defined as*

$$M(\sigma, f) = \sup_{t \in R} |f(\sigma + it)|, \qquad \sigma < \sigma_c^f,$$

then

$$\lim_{\sigma \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log L(\sigma, f)}{\sigma} \leqq \begin{matrix} \varrho \\ \lambda \end{matrix}. \tag{1. 4}$$

But if f is of nonzero finite Ritt order ϱ, type τ and lover type v, then

$$\lim_{\sigma \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\mathrm{Log}(\sigma \cdot f)}{\sigma} \leqq \begin{matrix} \tau \\ v \end{matrix}. \tag{1.5}$$

PROOF. We have, from (1. 2),

$$L(\sigma,f)=\lim_{T\to+\infty}\frac{1}{2T}\int_{-T}^{T}\log|f(\sigma+it)|\,dt\leqq\log M(\sigma,f). \tag{1. 6}$$

Therefore

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{L(\sigma,f)}{e^{\varrho\sigma}}\leqq\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log M(\sigma,f)}{e^{\varrho\sigma}} \tag{1. 7}$$

and

$$\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log L(\sigma,f)}{\sigma}\leqq\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log_2 M(\sigma,f)}{\sigma}, \tag{1. 8}$$

where $\log_2 x=\log\log x$. The result in (1. 4) and (1. 5) now follow from (1. 8) and (1. 7), respectively, since ([4], p. 77),

$$\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log_2 M(\sigma,f)}{\sigma}=\frac{\varrho}{\lambda},$$

and

$$\lim\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log M(\sigma,f)}{e^{\varrho\sigma}}=\frac{\tau}{\nu}.$$

2. Let $f\in E$ be an entire function of nonzero finite Ritt order ϱ whose logarithmic mean function is L. Since $\log L$ is an increasing convex function of σ, we may write

$$\log L(\sigma,f)=\log L(\sigma_0,f)+\int_{\sigma_0}^{\sigma}V(x,f)\,dx, \tag{2. 1}$$

where V is a real valued indefinitely increasing function of σ. We let

$$\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{V(\sigma,f)}{e^{\varrho\sigma}}=\frac{\alpha}{\beta}, \tag{2. 2}$$

and

$$\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log L(\sigma,f)}{e^{\varrho\sigma}}=\frac{p}{q}, \tag{2. 3}$$

where the constants $\alpha,\beta,p,q\in R_+^*\cup\{0\}$ (R_+^* is the set of extended positive reals), and prove the following results.

Theorem 3. *If L is the logarithmic mean function of an entire function $f\in E$ of nonzero finite Ritt order ϱ, and α, β, p, q are the constants defined as in* (2. 2) *and* (2. 3), *then*

$$\beta\leqq\varrho q\leqq\varrho p\leqq\alpha, \tag{2. 4}$$

$$\varrho p\geqq\frac{\alpha}{e}e^{\beta/\alpha}\geqq\beta, \tag{2. 5}$$

$$\varrho q\leqq\beta\left(1+\log\frac{\alpha}{\beta}\right)\leqq\alpha \tag{2. 6}$$

and

$$\alpha+\varrho q\leqq e\varrho p. \tag{2. 7}$$

PROOF. We choose a $k \in R_+ \cup \{0\}$ and get, from (2. 1),

$$\log L\left(\sigma+\frac{k}{\varrho}, f\right) = O(1)+\Big(\int_{\sigma_0}^{\sigma}+\int_{\sigma}^{\sigma+\frac{k}{\varrho}}\Big) V(x,f)\,dx. \tag{2. 8}$$

We first suppose that $\beta \in R_+$. Then, from (2. 2), we get

$$\frac{V(\sigma,f)}{e^{\varrho\sigma}} > \beta-\varepsilon,$$

for all $\sigma \geqq \sigma_0(\varepsilon, f)$ and for any $\varepsilon \in R_+$; whence, in view of (2. 8),

$$\frac{e^k \log L\left(\sigma+\frac{k}{\varrho}, f\right)}{e^{\varrho(\sigma+k/\varrho)}} > O(e^{-\varrho\sigma})+\frac{\beta-\varepsilon}{\varrho}(1-o(1))+\frac{k}{\varrho}\frac{V(\sigma,f)}{e^{\varrho\sigma}}. \tag{2. 9}$$

Taking inferior limits of both sides in (2. 9), as σ tends to plus infinity, we get

$$e^k q \geqq \frac{\beta}{\varrho}+\frac{k}{\varrho}\beta,$$

this, in turn, gives, for $k=0$,

$$\varrho q \geqq \beta, \tag{2. 10}$$

which also holds when $\beta=0$. If β were infinite, the above argument, with an arbitrarily large number instead of $\beta-\varepsilon$, gives $\varrho=+\infty$. Thus (2. 10) holds when $\beta \in R_+^* \cup \{0\}$. Further, from (2. 2), we get, for any $\varepsilon \in R_+$,

$$V(\sigma,f) < e^{\varrho\sigma}(\alpha+\varepsilon), \qquad \forall \sigma \geqq \sigma_0(\varepsilon, f). \tag{2. 11}$$

This estimate for V makes (2. 1) to yield

$$\frac{\log L(\sigma,f)}{e^{\varrho\sigma}} < O(e^{-\varrho\sigma})+\frac{\alpha+\varepsilon}{\varrho}(1-o(1)),$$

whence

$$\varrho p \leqq \alpha. \tag{2. 12}$$

Combining (2. 10) and (2. 12), since $\varrho p \geqq \varrho q$, we get (2. 4).

In order to establish (2. 5), we take superior limits, as σ tends to plus infinity, of both sides in (2. 9), and get

$$e^k p \geqq \frac{\beta+k\alpha}{\varrho},$$

which gives, for $k=1-\frac{\beta}{\alpha}$,

$$\varrho p \geqq \frac{\alpha}{e} e^{\beta/\alpha} \geqq \beta; \tag{2. 13}$$

the last inequality in (2. 13) follows from the fact that $x \geqq 0 \Rightarrow e^x \geqq ex$.

Further, from (2. 8) and (2. 11), we get

$$\log L\left(\sigma+\frac{k}{\varrho}, f\right) < O(1)+\frac{\alpha+\varepsilon}{\varrho}\left(e^{\varrho\sigma}-e^{\varrho\sigma_0}\right)+\frac{k}{\varrho} V\left(\sigma+\frac{k}{\varrho}, f\right),$$

or

$$\frac{e^k \log L\left(\sigma+\frac{k}{\varrho}, f\right)}{e^{\varrho(\sigma+k/\varrho)}} < O(e^{-\varrho\sigma})+\frac{\alpha+\varepsilon}{\varrho}(1-o(1))+\frac{k}{\varrho} \frac{V\left(\sigma+\frac{k}{\varrho}, f\right)}{e^{\varrho(\sigma+k/\varrho)}} e^k.$$

Taking inferior limits, as σ tends to plus infinity, of both sides, we get

$$e^k q \leqq \frac{\alpha+k\beta e^k}{\varrho},$$

which, on taking $k=\log(\alpha/\beta)$, gives

$$\varrho q \leqq \beta\left(1+\log\frac{\alpha}{\beta}\right) \leqq \alpha, \tag{2.14}$$

since $1+\log(\alpha/\beta) \leqq \exp(\log(\alpha/\beta))$. This proves (2. 6).

To prove (2. 7), we note that

$$V(\sigma, f) \leqq \varrho \int_{\sigma}^{\sigma+\frac{1}{\varrho}} V(x, f)\, dx,$$

since V is a positive indefinitely increasing function of σ. Adding $\varrho \log L(\sigma, f)$ on both sides of the above relation and using (2. 1) we get

$$\varrho \log L(\sigma, f)+V(\sigma, f) \leqq \varrho \log L\left(\sigma+\frac{1}{\varrho}, f\right).$$

Dividing throughout by $e^{\varrho\sigma}$ and proceeding to limits, we get (2. 7). This completes the proof of the theorem.

Remark. Since the function e^x-ex attains its minimum for $x=1$, in the relation (2. 5) actually $\beta<\frac{\alpha}{e} e^{\beta/\alpha}$ if $\alpha\neq\beta$, and in relation (2. 6) $\beta\left(1+\log\frac{\alpha}{\beta}\right) < \alpha$ if $\alpha\neq\beta$. Thus the equality in the relations (2. 5) and (2. 6) will occur only if $\alpha=\beta$. Moreover, from (2. 5),

$$\frac{\alpha}{e} e^{\beta/\alpha} \leqq \varrho p$$

or

$$\alpha \leqq e\varrho p. \tag{2.15}$$

A comparison of (2. 7) and (2. 15) shows that (2. 7) is a refinement of (2. 15).

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Theorem 4. *If $f \in E$ is an entire function of nonzero finite Ritt order ϱ, then* $\lim_{\sigma\to+\infty} \frac{V(\sigma,f)}{e^{\varrho\sigma}}$ *exists if, and only if,* $\lim_{\sigma\to+\infty} \frac{\log L(\sigma,f)}{e^{\varrho\sigma}}$ *exists, in which case*

$$\lim_{\sigma\to+\infty} \frac{V(\sigma,f)}{e^{\varrho\sigma}} = \varrho \lim_{\sigma\to+\infty} \frac{\log L(\sigma,f)}{e^{\varrho\sigma}}. \tag{2.16}$$

Proof. That if $\lim_{\sigma\to+\infty} \frac{V(\sigma,f)}{e^{\varrho\sigma}}$ exists, then $\lim_{\sigma\to+\infty} \frac{\log L(\sigma,f)}{e^{\varrho\sigma}}$ exists follows from (2. 4). We, therefore, suppose that

$$\lim_{\sigma\to+\infty} \frac{\log L(\sigma,f)}{e^{\varrho\sigma}} = p. \tag{2.17}$$

First let $p \in R_+$; then, for any $\varepsilon \in R_+$ and $\sigma > \sigma_1(\varepsilon)$,

$$(p-\varepsilon)e^{\varrho\sigma} < \log L(\sigma,f) < (p+\varepsilon)e^{\varrho\sigma}. \tag{2.18}$$

Hence, for any $\delta \in]0, 1[$, we have

$$\int_{\sigma}^{\sigma+\delta} V(x,f)\,dx = \int_{\sigma_0}^{\sigma+\delta} V(x,f)\,dx - \int_{\sigma_0}^{\sigma} V(x,f)\,dx = \log L(\sigma+\delta,f) - \log L(\sigma,f) <$$

$$< (p+\varepsilon)e^{(\sigma+\delta)\varrho} - (p-\varepsilon)e^{\sigma\varrho} =$$

$$= p(e^{\delta\varrho}-1)e^{\sigma\varrho} + \varepsilon(e^{\delta\varrho}+1)e^{\sigma\varrho} = p(\delta\varrho+\cdots)e^{\sigma\varrho} + \varepsilon(2+\delta\varrho+\cdots)e^{\sigma\varrho}.$$

Therefore, since V is an increasing function of σ,

$$\delta V(\sigma,f) \leqq \int_{\sigma}^{\sigma+\delta} V(x,f)\,dx < p(\delta\varrho+\cdots)e^{\sigma\varrho} + \varepsilon(2+\delta\varrho+\cdots)e^{\sigma\varrho}.$$

Since ε and δ are arbitrary, it follows that

$$\limsup_{\sigma\to+\infty} \frac{V(\sigma,f)}{e^{\varrho\sigma}} \leqq \varrho p. \tag{2.19}$$

Proceeding in a similar way we easily get

$$\int_{\sigma-\delta}^{\sigma} V(x,f)\,dx > p(\delta\varrho-\cdots)e^{\sigma\varrho} - \varepsilon(2-\delta\varrho+\cdots)e^{\sigma\varrho},$$

for any $\varepsilon \in R_+$, $\sigma > \sigma_2(\varepsilon)$ and any $\delta \in]0, 1[$.

Or, since V is an increasing function of σ,

$$\delta V(\sigma,f) \geqq \int_{\sigma-\delta}^{\sigma} V(x,f)\,dx > p(\delta\varrho-\cdots)e^{\sigma\varrho} - \varepsilon(2-\delta\varrho+\cdots)e^{\sigma\varrho},$$

whence, since ε, δ are arbitrary,

$$\liminf_{\sigma\to+\infty} \frac{V(\sigma,f)}{e^{\varrho\sigma}} \geqq \varrho p. \tag{2.20}$$

Thus, for $p \in R_+$, the relations (2. 19) and (2. 20) give us

(2.21)
$$\lim_{\sigma \to +\infty} \frac{V(\sigma, f)}{e^{\varrho\sigma}} = \varrho p.$$

If $p=0$, then (2. 19) gives

$$\lim_{\sigma \to +\infty} \frac{V(\sigma, f)}{e^{\varrho\sigma}} = 0,$$

and if $p = +\infty$, then taking an arbitrary large number M in place of $(p-\varepsilon)$ and proceeding as above we get

$$\lim_{\sigma \to +\infty} \frac{V(\sigma, f)}{e^{\varrho\sigma}} = +\infty.$$

Thus in each case $\lim\limits_{\sigma \to +\infty} \dfrac{V(\sigma, f)}{e^{\varrho\sigma}}$ exists.

The equality (2. 16) follows from (2. 17) and (2. 21).

References

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