On a class of associative functions

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Introduction

Since their introduction by MENGER in 1942, certain associative functions called t-norms have been studied not only in connection with their original application to probabilistic metric spaces, but also, largely through the work of SCHWEIZER and SKLAR, in connection with semigroup theory and functional equations.

In Section 1 of this paper*), we record results of an elementary nature, illustrating the strength of the associativity property in determining values assumed by t-norms. Section 2 consists of similar results, arising from solutions to certain conditionings of the Schröder functional equation, and we obtain representations for certain t-norms. In Section 3 an inversion operator is introduced into the class of functions defined on the unit square, and we are interested mainly in characterizing those t-norms whose inversions are t-norms. Again, we obtain representations for certain classes of t-norms.

We continue this introduction with definitions and preliminary results:

1 Definition. A *triangle norm* (briefly, a t-norm) is a two-place function T from the closed unit square $[0, 1]\times[0, 1]$ (henceforth written I^2) to the closed unit interval $[0, 1]$ which satisfies the following conditions:

i) $T(0, 0)=0$, $T(a, 1)=a$ (Boundary conditions)
ii) $T(a, b)\leqq T(c, d)$ whenever $a\leqq c$ and $b\leqq d$ (Monotonicity)
iii) $T(a, b)=T(b, a)$ (Symmetry)
iv) $T\big(T(a, b), c\big)=T\big(a, T(b, c)\big)$ (Associativity)

1a Definition. A t-norm is *continuous* if it is continuous in each place.

1b Definition. A t-norm is *strict* if it is continuous and, over the interior of I^2, strictly increasing in each place.

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1c Definitions. We define certain basic t-norms T_w, T_m, Prod, and Min, as follows:

$$T_w(a, b) = \begin{cases} a, & b = 1 \\ b, & a = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$T_m(a, b) = \max(a+b-1, 0)$$

$$\operatorname{Prod}(a, b) = ab$$

$$\operatorname{Min}(a, b) = \begin{cases} a, & a \leqq b \\ b, & b \leqq a \end{cases}$$

Of these, only Prod is a strict t-norm.

By part iv) of Definition 1, we can write $T(a, b, c)$ for the common value

$$T(T(a, b), c) = T(a, T(b, c)),$$

and, in general, for any points $x_1, \ldots, x_n$ in $[0, 1]$, we define

$$T(x_1, \ldots, x_n) = T(x_1, T(x_2, \ldots, x_n))$$

inductively. In particular, we define here the functions

$$T_1(x) = T(1, x)$$

$$T_n(x) = T\underbrace{(x, \ldots, x)}_{n\text{ places}}, \qquad n = 2, 3, \ldots .$$

If T is such that T_n has an inverse (e.g., when T is strict), we define

$$T_{-n} = T_n^{-1}, \qquad n = 1, 2, \ldots .$$

1d Definition. A t-norm T is *archimedean* if, for every $x \in (0, 1)$, $\lim_{n \to \infty} T_n(x) = 0$.

To prepare for the next definition we remark first that, associated with each t-norm is an abelian semigroup $([0, 1], T)$. That is, an associative, commutative binary operation T is defined for all $a, b \in [0, 1]$ by the rule $aTb = T(a, b)$. Clearly one endpoint, 0, is an annihilator, and the other endpoint, 1, is an identity. Such a semigroup is called an "(I)-semigroup" in [5] and a "thread" in [8].

If T is a t-norm and $[\lambda, \lambda']$ is a closed subinterval of $[0, 1]$, then T can be "shrunk" to a function T_λ defined on $[\lambda, \lambda'] \times [\lambda, \lambda']$ by

$$T_\lambda(x, y) = \lambda + (\lambda' - \lambda) T\left(\frac{x-\lambda}{\lambda'-\lambda}, \frac{y-\lambda}{\lambda'-\lambda}\right).$$

Then $([\lambda, \lambda'], T)$ is an (I)-semigroup. The motivation for the following definition is that, given a collection $\{T_\lambda\}$ of t-norms and a partition

$$\{[\lambda_a, \lambda_a']\}_{a \in A}$$

$$\{[\lambda_a, \lambda_a']\}_{a \in A}$$ ups

of $[0, 1]$, we can in many ways piece derived (I)-semigroups

$$([\lambda_a, \lambda_a'], T_{\lambda_a})$$

together to form a t-norm.

1e Definition ([4], p. 206). Let A be a totally ordered set and $\{S_a, \sigma_a\}_{a\in A}$ a collection of pairwise disjoint semigroups (S_a, σ_a) indexed by A. Then the *ordinal sum* of $\{(S_a, \sigma_a)\}_{a\in A}$ is the set $\bigcup\limits_{a\in A} S_a$ under the following binary operation σ:

$$x\sigma y = \begin{cases} x\sigma_a y & \text{if for some } a\in A, \text{ both } x \text{ and } y \text{ lie in } S_a \\ x & \text{if } x\in S_a \text{ and } y\in S_b \text{ for some } a, b\in A,\ a<b \\ y & \text{if } x\in S_a \text{ and } y\in S_b \text{ for some } a, b\in A,\ a>b. \end{cases}$$

It is immediate that (S, σ) is a semigroup.

Suppose now that a t-norm T is known to be an ordinal sum of (I)-semigroups and one-point semigroups. Let S be the set of idempotents of T. That is,

$$S = \{x\in[0, 1]: T_2(x) = x\}.$$

If S is not nowhere dense, let Q be the union of all open intervals covered by the closure of S. Define $S_1 = S\cap Q$. Then if $T_2(x)<x$ for some $x\in Q$, let $x'\in(T_2(x), x)\cap \cap S_1$. Then $T_2(x')=x'>T_2(x)$, contrary to the monotonicity property of t-norms, since $x'<x$. This shows that for $x\in Q$, $T_2(x)=x$, hence $S_1=Q$ and $Q\subset S$. We may therefore represent S as a union of an open set Q with a nowhere dense set Q'. It follows that, given a representation

$$\{([\lambda_a, \lambda_a'], T_{\lambda_a})\}_{a\in A}$$

for T, in which the underlying sets $[\lambda_a, \lambda_a']$ are intervals or singletons $\lambda_a=\lambda_a'$, we may give a clearer representation (however, with regard to the disjointness of the semigroups in Definition 1e, we here ignore common endpoints of intervals):

$$\{([\lambda_i, \lambda_i'], T_{\lambda_i})\}_{i\in N}\cup\{(Q', \mathrm{Min})\},$$

where N is a countable indexing set (each interval contains in its interior a rational number not contained in any other of the intervals) and Q' is a nowhere dense set. Q' can be, as in the Cantor set, uncountable.

Clearly, from Definition 1e, over that portion of I^2 lying outside the diagonalized subsquares $[\lambda_i, \lambda_i']\times[\lambda_i, \lambda_i']$, T coincides with Min.

We wish next to quote ([4], p. 206), where a theorem originating in ([5], p. 130) is phrased in terms of t-norms. First we need the definition given in [6] for an archimedean semigroup. Let J be a closed interval $[a, b]$ of the extended real line and let $S: J\times J \to J$ be an associative function satisfying the following conditions: (1) S is continuous, (2) S is nondecreasing in each place, (3) the endpoint b is a left unit, i.e., $S(b, x)=x$ for all $x\in J$, and (4) for all $x\in(a, b)$, $S(x, x)<x$. Then the semigroup (J, S) is archimedean. The theorem just mentioned above states, "Every continuous t-norm is either the t-norm Min, or is an ordinal sum of archimedean semigroups and one-point semigroups."

This theorem generalizes the theorem of Climescu [2] on the ordinal sum of two semigroups, which was applied in [7] to t-norms. Our principal interest here, however, is that, by this theorem, the discussion just preceding it and the representation obtained there now apply to all continuous t-norms.

2 Theorem (Aczél, [6], p. 170). *If T is a strict t-norm, then there exists a function F, defined, continuous, and strictly decreasing on* $[0, \infty)$, *with* $F(0)=1$ *and* $\lim\limits_{x\to\infty} F(x)=0$, *such that for every* $(a, b)\in(0, 1]\times(0, 1]$,

$$T(a, b) = F(F^{-1}(a)+F^{-1}(b)).$$

From

$$T(0, b) = \lim_{x\to 0} F(F^{-1}(x)+F^{-1}(b)) = T(b, 0) = 0,$$

we see that the function F, henceforth called a *generator* for T, completely determines T.

2a Theorem ([6], p. 171). *If T is a strict t-norm with generators F_1 and F_2, then there exists a real number $\lambda \neq 0$ such that $F_2(x) = F_1(\lambda x)$, $x \in [0, \infty)$. Conversely, if F is any generator of T and $\lambda \neq 0$, then $F(\lambda x)$ defines a generator for T.*

2b Theorem ([6], p. 171). *If F is a function defined, continuous, and strictly decreasing on $[0, \infty)$, with $F(0)=1$ and $\lim_{x\to\infty} F(x)=0$, then the two-place function T defined on $(0, 1]\times(0, 1]$ by*

$$T(a, b) = F(F^{-1}(a)+F^{-1}(b))$$

and extended to all of I^2 by continuity, is a strict t-norm.

1. Elementary results

3 Lemma. *Let T be a t-norm. If the values of $T(x, y)$ are determined for fixed $x \in (0, 1)$ and all $y \in (0, x)$, then the values of $T(T_n(x), y)$ are determined for all $y \in [0, 1]$, $n=2, 3, \ldots$.*

Proof. Under the hypothesis, the values of $T(x, T(x, y))$, and hence by associativity, those of $T(T(x, x), y)$, are known for all $y \in [0, 1]$. Suppose, for arbitrary $n > 2$, that the values of $T(T_{n-1}(x), y)$ are known for all $y \in [0, 1]$. Then the lemma follows inductively from

$$T(T_n(x), y) = T(x, T(T_{n-1}(x, y))).$$

4 Theorem. *Let T be a continuous t-norm. If the values of $T(x, y)$ are determined in a region $x_1 \leqq x \leqq x_2$, $0 < y < x$, then the values of $T(x, y)$ are determined in each region*

$$[T_n(x_1), T_n(x_2)]\times[0, 1], \qquad n=2, 3, \ldots .$$

Moreover, if T is also archimedean and $T_2(x_2) \geqq x_1$, then the values of $T(x, y)$ are determined for $0 \leqq x \leqq x_1$, $0 \leqq y \leqq 1$.

Proof. Let $n \geqq 2$. Since T is continuous, T_n is continuous. Thus, if

$$x' \in [T_n(x_1), T_n(x_2)],$$

there must exist $x \in [x_1, x_2]$ for which $T_n(x) = x'$. Then

$$T(x', y) = T(T_n(x), y),$$

so that by Lemma 3, $T(x', y)$ is determined for all $y \in [0, 1]$.

Now suppose that, in addition to being continuous, T is archimedean and satisfies $T_2(x_2) \geqq x_1$. Then the strip

$$[T_2(x_1), T_2(x_2)] \times [0, 1]$$

meets the strip

$$[x_1, x_2] \times [0, 1].$$

For arbitrary $n > 2$, suppose the strip

$$[T_n(x_1), T_n(x_2)] \times [0, 1]$$

meets the strip

$$[T_{n-1}(x_1), T_{n-1}(x_2)] \times [0, 1].$$

Then $T_n(x_2) \geqq T_{n-1}(x_1)$, so that

$$T_{n+1}(x_2) \geqq T(T_n(x_2), x_1) \geqq T(T_{n-1}(x_1)) = T_n(x_1).$$

Thus, the strips

$$[T_{n+1}(x_1), T_{n+1}(x_2)] \times [0, 1]$$

and $[T_n(x_1), T_n(x_2)] \times [0, 1]$ meet. Since

$$\lim_{n \to \infty} T_n(x_1) = 0,$$

the strips

$$[T_n(x_1), T_n(x_2)] \times [0, 1], \qquad n = 2, 3, \dots$$

cover $[0, x_1] \times [0, 1]$.

5 Theorem. *Let D be a subset of the unit square I^2 such that, for each $\delta \in (0, 1)$, there exists $x_\delta \in (0, 1)$ such that for each $x \geqq x_\delta$, $\{x\} \times (0, \delta) \subset D$. Let T be an archimedean and continuous t-norm. Then T is determined by its values in D.*

Proof. Let $x_0, \delta \in (0, 1)$. By the lemma immediately below, there exist $x \in (x_0, 1)$ and n such that $T_n(x) = x_0$ and $\{x\} \times [0, \delta) \subset D$. Then

$$T(x_0, y) = T(T_n(x), y) = T(x, T(T_{n-1}(x), y))$$

is determined for all y satisfying

$$0 \leqq T(T_{n-1}(x), y) < \delta,$$

hence certainly for all $y \in [0, \delta)$. Since x_0 and δ were arbitrarily chosen, and since the values of $T(x_0, 1)$ are prescribed, the proof is complete.

Proof *of the lemma*. For each positive integer n, define x_n by $T_n(x_n) = x_0$. The existence of this x_n is ascertained by applying the intermediate value theorem to the continuous function T_n. Now $\lim_{x \to \infty} x_n = 1$, for if this limit were some $\varepsilon < 1$, then for all n, $T_n(x_n) \leqq T_n(\varepsilon)$, so that

$$x_0 = \lim_{n \to \infty} T_n(x_n) \leqq \lim_{n \to \infty} T_n(\varepsilon) = 0.$$

contrary to $x_0 \in (0, 1)$. Thus if x_δ is chosen so that for $x \in [x_\delta, 1]$, $\{x\} \times (0, \delta) \subset D$, then we can choose n so large that $x_\delta < x_n < 1$ and have $\{x_n\} \times (0, \delta) \subset D$.

6 Theorem. *Let $x_1 \in (0, 1)$ and let N be a positive integer. Then every strict t-norm T is determined by its values $T_n(x)$, $x \in [x_1, 1]$, $n = N, N+1, \ldots$.*

PROOF. Suppose $T_n(x)$ is known for all $x \in [x_1, 1]$, $n = N, N+1, \ldots$. Let f be that generator for T which satisfies $f(1) = x_1$. (If F is any generator for T, then $F(r) = x_1$ for some $r > 0$; our f is then the generator given by $f(x) = F(rx)$.) Now

$$T_n(x_1) = f(nf^{-1}(x_1)) = f(n), \qquad n = 1, 2, \ldots .$$

Let x_2 be that number which satisfies $T_N(x_2) = x_1$. Then since

$$f(Nf^{-1}(x_2)) = x_1 = f(1),$$

we have $f^{-1}(x_2) = \dfrac{1}{N}$, so that

$$T_n(x_2) = f(nf^{-1}(x_2)) = f\left(\frac{n}{N}\right), \qquad n = 1, 2, \ldots .$$

Suppose for arbitrary $k > 1$ that x_{k-1} has been defined so that

$$T_n(x_{k-1}) = f\left(\frac{n}{N^{k-2}}\right), \qquad n = 1, 2, \ldots .$$

Let x_k be that number which satisfies $T_N(x_k) = x_{k-1}$. Then

$$f(Nf^{-1}(x_k)) = x_{k-1} = f\left(\frac{1}{N^{k-2}}\right).$$

Hence $f^{-1}(x_k) = \dfrac{1}{N^{k-1}}$ and

$$T_n(x_k) = F(nf^{-1}(x_k)) = f\left(\frac{n}{N^{k-1}}\right), \qquad n = 1, 2, \ldots .$$

Thus, inductively, we obtain a sequence $x_1, x_2, \ldots$ with $\lim_{x \to \infty} x_k = 1$, and for $k = 1, 2, \ldots$, we find that the values $f\left(\dfrac{n}{N^{k-1}}\right)$ are determined for $n = 0, 1, \ldots$. The set

$$\left\{\frac{n}{N^{k-1}} : n, k \text{ are positive integers}\right\}$$

is dense in $[0, \infty)$, so that the continuous function f is determined on $[0, \infty)$; hence T is determined.

7 Theorem. *For any t-norm T, the functions T_n, $n = 2, 3, \ldots$, are determined by the functions T_p, $p = 2, 3, 5, 7, 11, \ldots$. Thus every strict t-norm T is determined by the functions T_p, p prime.*

PROOF. For the first assertion, let

$$n = p_1^{e_1} p_2^{e_2} \ldots p_m^{e_m}$$

be the prime factorization for n, and suppose that T_k has been shown to be determined for each $k < n$. Then the desired result follows from $T_n = T_{p_1} \circ T_{np_1}^{-1}$.

The second assertion now follows from Theorem 6.

2. Determinations of values by Schröder's equation

Here we deal with the relationships between certain prescribed functions q and the class of strict t-norms T such that $q=T_n$ for some n. In each of the theorems the method consists of recognizing a Schröder functional equation of a type solved in [3] and interpreting the solution in terms of t-norms.

In this section, we denote by q^k the k^{th} iterate of q, given by $q^1=q$ and

$$q^k = q\circ q^{k-1}, \qquad k=2,3,\dots .$$

The inverse $(q^k)^{-1}$ is written q^{-k}. Similarly, we employ the symbols

$$f^k, \qquad k=0,\pm 1,\pm 2,\dots .$$

Also we write, for example, $f\circ g\circ h(x)$ instead of $f(g(h(x)))$, for expressions of lengthy functional compositions.

8 Hypothesis. Let $s\in(0,1)$. Let f be a continuous strictly increasing function from $[0,\infty)$ into $[0,\infty)$ such that $f(0)=0$ and $f(x)<x$ for all $x\in(0,\infty)$.

8a Theorem ([3], p. 29). Assume Hypothesis 8. Then every function $\bar\varphi$ defined on an interval $[f(x_0), x_0]$, $x_0\in(0,\infty)$, and fulfilling the condition $\bar\varphi(f(x_0))=s\bar\varphi(x_0)$ can be uniquely extended to a solution φ of the Schröder equation $\varphi(f(x))=s\varphi(x)$ in $[0,\infty)$. If $\lim\limits_{x\to\infty} f(x)=\infty$, then φ is given by

$$\varphi(x) = \begin{cases} s^k\bar\varphi(f^{-k}(x)), & x\in(f^{k+1}(x_0), f^k(x_0)),\ k=0,\pm 1,\pm 2,\dots \\ 0, & x=0, \end{cases}$$

where $f^k=f\circ f^{k-1}$, $k=1,2,\dots$, and $f^k=(f^{-k})^{-1}$, $k=(-1,-,-2,\dots)$. Moreover, if $\bar\varphi$ is continuous, then φ is continuous.

8b Theorem ([3], p. 35). *The function φ in Theorem 8a is strictly monotonic on $[0,\infty)$.*

8c Theorem ([3], p. 36). *Assume Hypothesis 8. Then every solution φ to the Schröder equation $\varphi(f(x))=s\varphi(x)$ which is strictly monotonic in a (right) neighborhood of 0 is strictly monotonic in the whole of $[0,\infty)$.*

8d Theorem ([3], p. 36). *Assume Hypothesis 8, and, in addition, assume for some $p\geqq 2$ that $f\in C^p$, i.e., f has a continuous p^{th} order derivative over $[0,\infty)$. Further assume that $f'(0+)=s$, and that $f'(x)>0$ for all $x\in(0,\infty)$. Then for every real number d there exists one and only one C^p solution φ to the equation $\varphi(f(x))=s\varphi(x)$ in $[0,\infty)$ which fulfills the condition $\varphi'(0)=d$. This solution is given by the formula*

$$\varphi(x) = d\lim_{k\to\infty} s^{-k}f^k(x) \qquad (f^k \text{ as in Theorem 8a}).$$

8e Theorem ([3], p. 40). *Assume Hypothesis 8, and, in addition, assume that $\lim\limits_{x\to 0+} f'(x)=s$ (f, being strictly increasing, is differentiable almost everywhere) and* *that f is convex in $(0, \infty)$. Then the Schröder equation $\varphi(f(x))=s\varphi(x)$ possesses a unique one-parameter family of convex solutions in $(0, \infty)$. These solutions are given by*

$$\varphi(x) = d \lim_{k\to\infty} \frac{f^k(x)}{f^k(c)}, \qquad (f^k \text{ as in Theorem 8a})$$

where c is an arbitrary fixed point from $(0, \infty)$, and d is an arbitrary constant.

9a Lemma. *Let the hypothesis of Theorem 9 be assumed, and let $h(y) = \dfrac{1}{y} - 1$. Then the function*

$$f(x) = h \circ q^{-1} \circ h^{-1}(x)$$

satisfies the conditions on the function f of Theorem 8a. If G is a solution to the equation

$$G(f(x)) = \frac{1}{n} G(x), \qquad x \in [0, \infty),$$

obtained as an extension of a function $\bar{G}$ defined, continuous, and strictly increasing on an interval as in Theorem 8a, then the function

$$F(x) = h^{-1}(G^{-1}(x))$$

is a continuous, strictly decreasing solution to our equation

$$q(x) = F(nF^{-1}(x)), \qquad x \in [0, \infty).$$

Proof. It is easy to verify that $f(x)$ satisfies the conditions on the function f of Theorem 8a, and we omit this verification. (A more general proposition is proved in Lemma 11c.) For the second assertion, we have

$$G \circ h \circ q^{-1} \circ h^{-1}(x) = \frac{1}{n} G(x), \qquad x \in [0, \infty)$$

$$G \circ h \circ q^{-1}(x) = \frac{1}{n} G \circ h(x), \qquad x \in (0, 1]$$

$$H(q^{-1}(x)) = \frac{1}{n} H(x), \qquad x \in [0, 1],\ H(x) = G(h(x)).$$

Now by Theorem 8b, G is strictly increasing. It follows that H is strictly decreasing. Thus, upon inverting, we obtain

$$H^{-1}(nx) = q(H^{-1}(x)), \quad x \in [0, \infty), \qquad H^{-1}(x) = h^{-1}(G^{-1}(x))$$

and see that H^{-1} has the desired properties.

9b Lemma. Let $f(x)$ be as in Lemma 9a. Then

$$f^k(x) = h \circ q^{-k} \circ h^{-1}(x), \qquad k = 0, \pm 1, \pm 2, \ldots,\ x \in [0, \infty).$$

9 Theorem. *Suppose q is a continuous, strictly increasing function on* $[0, 1]$ *with $q(0)=0$, $q(1)=1$, and $q(x)<x$ for all $x \in (0, 1)$. Let $x_0 \in (0, \infty)$, and suppose that for some $n \geqq 2$, $\bar{F}$ is a continuous, strictly decreasing function from $\left[\dfrac{x_0}{n}, x_0\right]$ into*

(0, 1) *satisfying* $\bar{F}(x_0)=q\left(\bar{F}\left(\frac{x_0}{n}\right)\right)$. *Then* $\bar{F}$ *has a unique extension* F *to* $[0, \infty)$ *satisfying* $q(y)=F(nF^{-1}(y))$ *for all* $y\in(0, 1]$. *This function is given by*

$$F(x) = \begin{cases} q^{-k}(\bar{F}(n^k x)) & \text{for } x\in\left[\frac{x_0}{n^{k+1}}, \frac{x_0}{n^k}\right], \ k=0, \pm 1, \dots \\ 1 & \text{for } x=0, \end{cases}$$

and is a generator. If T *is the t-norm generated by* F, *then* $q=T_n$.

PROOF. Suppose $\bar{F}$ is a continuous, strictly decreasing function on $\left[\frac{x_0}{n}, x_0\right]$ into (0, 1) satisfying

$$\bar{F}(x_0) = q\left(\bar{F}\left(\frac{x_0}{n}\right)\right)$$

for some $x_0\in(0, \infty)$ and $n\geqq 2$. Define

$$\bar{G}(x) = \bar{F}^{-1}\circ h^{-1}(x),$$

with h as in Lemma 9a. First we shall argue that $\bar{G}$ is a continuous, strictly increasing function on $[h\circ q^{-1}\circ h^{-1}(y_0), y_0]$, where $y_0 = h\circ\bar{F}(x_0)$: Since the domain of $\bar{F}$ is $\left[\frac{x_0}{n}, x_0\right]$, that of $\bar{F}^{-1}$ is $\left[\bar{F}(x_0), \bar{F}\left(\frac{x_0}{n}\right)\right]$. Hence the domain of $\bar{F}^{-1}\circ h^{-1}$ is

$$\left[h\circ\bar{F}\left(\frac{x_0}{n}\right), h\circ\bar{F}(x_0)\right],$$

which we can re-write as

$$\left[h\circ\bar{F}\left(\frac{x_0}{n}\right), h\circ q\circ\bar{F}\left(\frac{x_0}{n}\right)\right],$$

hence as

$$[h\circ q^{-1}\circ h^{-1}(y_0), y_0]$$

where $y_0 = h\circ\left(q\circ\bar{F}\left(\frac{x_0}{n}\right)\right) = h\circ\bar{F}(x_0)$.

Clearly, $\bar{G}$ is continuous and strictly increasing.

Next, we shall see that $\bar{G}$ satisfies

$$\bar{G}(h\circ q^{-1}\circ h^{-1}(y_0)) = \frac{1}{n}\bar{G}(y_0):$$

$$q\circ\bar{F}\left(\frac{x_0}{n}\right) = \bar{F}(x_0)$$

$$h^{-1}(y_0) = \bar{F}(n\bar{F}^{-1}\circ q^{-1}\circ h^{-1}(y_0))$$

$$\frac{1}{n}F^{-1}\circ h^{-1}(y_0) = \bar{F}\circ q^{-1}\circ h^{-1}(y_0)$$

$$\frac{1}{n}\bar{G}(y_0) = \bar{F}^{-1}\circ h^{-1}\circ h\circ q^{-1}\circ h^{-1}(y_0) = \bar{G}(h\circ q^{-1}\circ h^{-1}(y_0)).$$

One more item must be checked before we can apply Theorem 8a, namely that $\lim_{x\to\infty} f(x)=\infty$, but this follows immediately from $\lim_{x\to 0} q^{-1}(x)=0$.

Now, Theorem 8a and Theorem 8b apply. Let G be the unique continuous, strictly increasing extension of $\bar{G}$ to $[0,\infty)$ satisfying

$$G(h\circ q^{-1}\circ h^{-1}(y)) = \frac{1}{n}G(y)$$

for all $y\in[0,\infty)$. Then G is given by

$$G(y) = \begin{cases} \left(\frac{1}{n}\right)^k \bar{G}(f^{-k}(y)) & \text{for } y\in(f^{k+1}(y_0), f^k(y_0)), \\ & \quad k=0,\pm 1,\pm 2,\dots \\ 0 & \text{for } y=0, \end{cases}$$

where $f(y) = h\circ q\circ h^{-1}(y)$. Applying Lemma 9b,

$$G(y) = \begin{cases} \left(\frac{1}{n}\right)^k \bar{G}\big(h\circ q^k\circ h^{-1}(y)\big) & \text{for } y\in[h\circ q^{-k-1}\circ h^{-1}(y_0), h\circ q^{-k}\circ h^{-1}(y_0)], \\ & \quad k=0,\pm 1,\pm 2,\dots \\ 0 & \text{for } y=0. \end{cases}$$

Equivalently, for $w\in(0,1)$, we have

$$G\big(h(w)\big) = \begin{cases} \left(\frac{1}{n}\right)^k \bar{G}\big(h\circ q^k(w)\big) & \text{for } h(w)\in\big[h(q^{-k-1}(w_0)), h(q^{-k}(w_0))\big], \\ & \quad w_0=h^{-1}(y_0),\ k=0,\pm 1,\pm 2,\dots \\ 0 & \text{for } w=1, \end{cases}$$

so that for $F^{-1}(y)=G\big(h(y)\big)$ and $\bar{F}^{-1}(y)=\bar{G}\big(h(y)\big)$, we have

$$F^{-1}(w) = \begin{cases} \left(\frac{1}{n}\right)^k \bar{F}^{-1}\big(q^k(w)\big) & \text{for } w\in[q^{-k}(w_0), q^{-k-1}(w_0)], \\ & \quad k=0,\pm 1,\pm 2,\dots \\ 0 & \text{for } w=1. \end{cases}$$

Now for $w\in(0,1)$, the substitution $w=F(x)$ leads to

$$n^k x = \bar{F}^{-1}\big(q^k(F(x))\big), \quad k=0,\pm 1,\pm 2,\dots, \tag{1}$$

hence to $F(x)=q^{-k}\big(\bar{F}(n^k x)\big)$, $k=0,\pm 1,\pm 2,\dots$.

It remains to be seen that for $w\in[q^{-k}(w_0), q^{-k-1}(w_0)]$, we have

$$x\in\left[\frac{x_0}{n^{k+1}}, \frac{x_0}{n^k}\right]: \text{From } F(x) = w,$$

we have

$$x\in[F^{-1}\circ q^{-k}\circ h^{-1}(y_0), F^{-1}\circ q^{-k-1}\circ h^{-1}(y_0)].$$

That is,

$$x\in[F^{-1}\circ q^{-k}\circ \bar{F}(x_0), F^{-1}\circ q^{-k-1}\circ \bar{F}(x_0)].$$

By equation (1) above, this last interval is just

$$\left[\frac{x_0}{n^{k+1}}, \frac{x_0}{n^k}\right], \qquad k=0, \pm 1, \pm 2, \ldots .$$

The method of proof shows that the extension F of $\bar{F}$ is unique, with respect to the required properties, since this uniqueness is equivalent to that of the function φ in Theorem 8a.

To ascertain that F is a generator, we must see that $\lim_{x\to\infty} F(x)=0$: We have for each integer k and each $x_0>0$,

$$F\left(\frac{x_0}{n^k}\right) = q^{-k}(\bar{F}(x_0)),$$

so that what we wish to establish is that

$$\lim_{k\to-\infty} F\left(\frac{x_0}{n^k}\right) = \lim_{k\to\infty} q^k(\bar{F}(x_0)) = 0.$$

This must be the case, since $q(x)<x$ for all $x\in(0,1)$: otherwise let x_∞ be the supposed positive limit point of the (decreasing) sequence $q^k(\bar{F}(x_0))$. Since q is continuous,

$$q(x_\infty) = q\big(\lim_{k\to\infty} q^k(\bar{F}(x_0))\big) = \lim_{k\to\infty} q^{k+1}(\bar{F}(x_0)) = x_\infty,$$

a contradiction.

The last statement in the theorem is a re-wording of the Schröder equation $q(x)=F(nF^{-1}(x))$, since $T_n(x)=F(nF^{-1}(x))$.

9c Corollary. Let q be a continuous strictly increasing function on $[0, 1]$ with $q(0)=0$, $q(1)=1$, and $q(x)<x$ for all $x\in(0,1)$. Then for any positive integer $n\geqq 2$, there exist strict t-norms T such that $T_n=q$.

Proof. Let such q and n be given. Let $x_0\in(0,\infty)$. Let y be any number in $(0,1)$. Then $q(y)<y$, and we may define in many ways a continuous function $\bar{F}$ on $\left[\frac{x_0}{n}, x_0\right]$, strictly decreasing from $y=\bar{F}\left(\frac{x_0}{n}\right)$ to $q(y)=q\left(\bar{F}\left(\frac{x_0}{n}\right)\right)=\bar{F}(x_0)$. By Theorem 9, any such $\bar{F}$ can be extended to a generator F whose t-norm T satisfies $T_n=q$.

If $\bar{F}_1$ and $\bar{F}_2$ are two distinct functions on $\left[\frac{x_0}{n}, x_0\right]$ obtained as above, then clearly F_2 cannot be related to $\bar{F}_1$ by $\bar{F}_2(x)=\bar{F}_1(\lambda x)$ for any constant λ. Thus by Theorem 2a, the extensions F_1 and F_2 generate distinct strict t-norms.

The corollary shows, for example, that many t-norms T coincide with Prod over the diagonal $\{(x,x): x\in[0,1]\}$ of I^2, i.e., $T_2(x)=x^2$ for many t-norms T.

More generally, for any strict t-norm T and $n\geqq 2$, a large class of strict t-norms T' satisfy $T'_n=T_n$. We shall show, however, that under certain conditions on the prescribed function q, there is one and only one strict t-norm of some large class (e.g., having generator in C^p) which satisfies $T_n=q$.

10a Lemma. *Assume the hypothesis of Theorem 10. Let h be as in Lemma 9a, and let $f = h\circ q^{-1}\circ h^{-1}$. This function satisfies the conditions on the function f of*

Theorem 8d with $s=\frac{1}{n}$, *and if G is that solution to the equation* $G(f(x))=\frac{1}{n}G(x)$, $x\in[0,\infty)$ *with* $G'(0)=d\neq 0$, *obtained by Theorem 8d, then the function F given by* $F(x)=h^{-1}(G^{-1}(x))$ *is that* (C^p, if $q\in C^p$) *solution to our equation* $q(x)=F(nF^{-1}(x))$ *which satisfies* $F'(0)=-\frac{1}{d}$.

Proof. The first assertion can be routinely verified, and its proof is included in that of Lemma 11c. In proving Lemma 9a, we already showed that $h^{-1}(G^{-1}(x))$ solves the equation $q(x)=F(nF^{-1}(x))$. If $q\in C^p$, then by Theorem 8d, $G\in C^p$, so that $F\in C^p$. If $G'(0)=d$, it is easily checked that $F'(0)=-\frac{1}{d}$. Finally, it is clear that the uniqueness of F is equivalent to that of G, the latter being given by Theorem 8d.

10b Lemma. *Suppose* $\{G_k\}_{k=0}$ *is a sequence of strictty increasing functions from* $[0,\infty)$ *onto* $[0,\infty)$ *which converges pointwise to* G_0. *Then* $\{G_k^{-1}\}_{k=0}$ *converges pointwise to* G_0^{-1}.

10c Lemma. *Assume the hypothesis of Theorem 10. Let* $d>0$. *Define* $G_k(x)=dn^k h\circ q^{-k}\circ h^{-1}(x)$ *for* $k=1,2,\ldots$ *and* $G_0(x)=\lim_{k\to\infty}G_k(x)$. *Then the sequence* $\{G_k\}_{k=0}$ *fulfills the hypothesis of Lemma 10b.*

Proof. Letting f be as in Lemma 10a, we have

$$G_k(x)=d\left(\frac{1}{n}\right)^{-k}f^k(x)$$

by Lemma 9b. By Theorem 8d, $\lim_{k\to\infty}G_k(x)$ exists for all $x\in[0,\infty)$ and defines there a solution $\varphi=G_0$ to the Schröder equation

$$\varphi(f(x))=\frac{1}{n}\varphi(x).$$

Since $d>0$, each G_k is easily seen to be strictly increasing and onto. To ascertain that these two properties hold also for G_0, we appeal first to Theorem 8c, which states that if G_0 is strictly increasing in a (right) neighborhood of 0, then G_0 is strictly increasing on all of $[0,\infty)$. Now $G_0'(0)=d>0$, so that as a pointwise limit of strictly increasing functions G_k, G_0 must be strictly increasing in some (right) neighborhood of 0. To see that G_0 is onto $[0,\infty)$, we note that G_0 can (in many ways) be obtained from a function $\bar\varphi$ as in Theorem 8a, and that the following argument therefore applies: Taking $\bar\varphi$ and x_0 so that $\bar\varphi(x_0)>0$, we have

$$G_0(f^k(x_0))=\varphi(f^k(x_0))=\left(\frac{1}{n}\right)^k\bar\varphi f^{-k}(f^k(x_0))=\left(\frac{1}{n}\right)^k\bar\varphi(x_0),$$

so that

$$\lim_{x\to\infty}G_0(x)=\lim_{k\to-\infty}G_0(f^k(x_0))=\lim_{k\to\infty}n^k\bar\varphi(x_0)=\infty.$$

Thus, as a continuous function, G_0 must assume all non-negative values.

10 Theorem. *Suppose q is a function which satisfies the following conditions:* (i) *For some $p \geqq 2$, $q \in C^p$ (the class of functions continuous on* $[0, 1]$ *with continuous* p^{th} *order derivative in* $(0, 1)$*)*; (ii) $q(0)=0$, $q(1)=1$; (iii) $q'(1)$ *is a positive integer* n; (iv) $q(x)<x$ *and* $q'(x)>0$ *for all* $x \in (0, 1)$. *Then there exists one and only one strict t-norm T, having generator in C^p, such that $T_n = q$. Moreover, a generator F for T is given by*

$$F(x) = \lim_{k \to \infty} q^k \left(\frac{n^k}{x + n^k} \right),$$

and T itself is given by

$$T(x, y) = \lim_{k \to \infty} q^k \big(\Omega (q^{-k}(x), q^{-k}(y)) \big),$$

where Ω is the strict t-norm given by

$$\Omega(a, b) = \frac{ab}{a+b-ab}.$$

PROOF. Upon applying Lemma 10a, we see that the unique C^p solution to

$$\varphi\big(f(x)\big) = \frac{1}{n} \varphi(x)$$

such that $\varphi'(0) = d$ is given by

$$\varphi(x) = d \lim_{k \to \infty} \left(\frac{1}{n} \right)^k f^k(x).$$

That is, letting

$$G_k(x) = dn^k h \circ q^{-k} \circ h^{-1}(x),$$

the function

$$G_0(x) = \lim_{k \to \infty} G_k(x)$$

is the unique C^p solution to

$$G_0\big(f(x)\big) = \frac{1}{n} G_0(x), \qquad x \in [0, \infty) \text{ and } G_0'(0) = d.$$

Now,

$$G_k^{-1}(x) = h \circ q^k \circ h^{-1} \left(\frac{x}{dn^k} \right),$$

so that

$$h^{-1}(G_k^{-1}(x)) = q^k \circ h^{-1} \left(\frac{x}{dn^k} \right) = q^k \left(\frac{1}{\dfrac{x}{dn^k} + 1} \right) = q^k \left(\frac{dn^k}{x + dn^k} \right),$$

and by Lemma 10b and Lemma 10c,

$$F(x) = \frac{1}{G_0^{-1}(x) + 1} = \frac{1}{\lim\limits_{k \to \infty} G_k^{-1}(x) + 1} = \lim_{k \to \infty} \frac{1}{G_k^{-1}(x) + 1} = \lim_{k \to \infty} q^k \left(\frac{dn^k}{x + dn^k} \right).$$

By Theorem 2a, the t-norm generated by F does not depend on d. Hence with $d = 1$, we obtain the desired representation for F. Recall that $F \in C^p$, since G_0, and hence G_0^{-1}, lie in C^p.

Now $F(x)=h^{-1}(G^{-1}(x))$, so

$$F^{-1}(x) = G(h(x)) = \lim_{k\to\infty} n^k \left(\frac{1}{q^{-k}(x)} - 1\right).$$

Thus $T(x, y) = F(F^{-1}(x)+F^{-1}(y))$. The desired expression for Ω is now easily obtained.

11a Definition. A function $q:[0, 1]\to[0, 1]$ is *co-convex* (relative to h) if there exists a strictly decreasing function h from $(0, 1]$ onto $[0, \infty)$ such that the function f given by $f(x) = h\circ q^{-1}\circ h^{-1}(x)$ is convex.

11b Lemma. *Assume the hypothesis of Theorem 11. Then the function f given in Definition 11a satisfies the conditions on the function f in Theorem 8e.*

11c Lemma. *Assume the hypothesis of Theorem 11. Let $c\in(0, \infty)$. Define*

$$G_k(x) = \frac{h\circ q^{-k}\circ h^{-1}(x)}{h\circ q^{-k}\circ h^{-1}(c)}, \qquad k=1, 2, \dots$$

and

$$G_0(x) = \lim_{k\to\infty} G_k(x).$$

Then the sequence $\{G_k\}_{k=0}$ fulfills the hypothesis of Lemma 10b.

Proof. Letting $f(x) = h\circ q^{-1}\circ h^{-1}(x)$, we have

$$G_k(x) = \frac{f^k(x)}{f^k(c)}.$$

By Lemma 11b, we can apply Theorem 8e, so that $\lim_{k\to\infty} G_k(x)$ exists for all $x\in[0, \infty)$ and defines there a solution $\varphi=G_0$ to the Schröder equation $\varphi(f(x))=\frac{1}{n}\varphi(x)$. Since

$$q^{-k}, \qquad k=1, 2, \dots,$$

is strictly increasing, we have for $0\leqq x_1<x_2$,

$$\begin{gathered} h^{-1}(x_1) > h^{-1}(x_2) \\ q^{-k}(h^{-1}(x_1) > q^{-k}(h^{-1}(x_2)) \\ h\circ q^{-k}\circ h^{-1}(x_1) < h\circ q^{-k}\circ h^{-1}(x_2) \\ \frac{f^k(x_1)}{f^k(c)} < \frac{f^k(x_2)}{f^k(c)}, \end{gathered}$$

which shows that f is strictly increasing. The remaining assertions are proved exactly as in Lemma 10c.

11 Theorem. *Suppose q is a continuous, strictly increasing, co-convex function on $[0, 1]$ with $q(0)=0$, $q(1)=1$, $q(x)<x$ for all $x\in(0, 1)$, and suppose further that $\lim_{k\to 1+} q'(x)$ is a positive integer n (since q is monotonic, it is differentiable almost everywhere). Then there exists one and only one strict t-norm T such that $q=T_n$.*

A generator F for T is given by

$$F(x) = \lim_{k\to\infty} q^k \circ h^{-1}\big[(h \circ q^{-k} \circ h^{-1}(c))x\big],$$

where h is any function relative to which q is co-convex, and c is any fixed point from $(0, \infty)$.

PROOF. Just as in the cases of Lemma 9a and Lemma 10a, if G is a convex solution to the equation

$$G(f(x)) = \frac{1}{n} G(x),$$

$x \in [0, \infty)$, obtained by Theorem 8e, then the function F given by

$$F(x) = h^{-1} \circ G^{-1}(x)$$

is the corresponding solution to our equation

$$q(x) = F(nF^{-1}(x)).$$

In the notation of Lemma 11c, we have by Theorem 8e,

$$G(x) = \lim_{k\to\infty} G_k(x),$$

and by Lemmas 11c and 10b,

$$G^{-1}(x) = \lim_{k\to\infty} G_k^{-1}(x).$$

Thus,

$$F(x) = h^{-1} \circ G^{-1}(x) = \lim_{k\to\infty} h^{-1} \circ G_k^{-1}(x) =$$

$$= \lim_{k\to\infty} h^{-1}\big(h \circ q^k \circ h^{-1}\big[(h \circ q^{-k} \circ h^{-1}(c))x\big]\big) = \lim_{k\to\infty} q^k \circ h^{-1}\big[(h \circ q^{-k} \circ h^{-1}(c))x\big].$$

We mention here two questions for further investigation. (1) If the prescribed function T_n is continuous and T is a t-norm satisfying

$$T(\underbrace{x, x, \ldots, x}_{n \text{ places}}) = T_n(x),$$

must T be continuous? (2) To what extent is a t-norm T determined by a prescribed function $q(x, y)$, x fixed and y variable from 0 to 1, if $T(x, y)$ is to be equal to $q(x, y)$ for such x and y? This second question should lead to a study of Abel's functional equation

$$\varphi(f(y)) = c + \varphi(y),$$

and, we suspect, to results which parallel Theorems 9, 10, and 11.

3. Inversion of t-norms

Inasmuch as the first two sections of this paper reveal one aspect of the associativity condition, namely its leading role, in conjunction with certain prescribed conditions, in determining t-norms, this present section reveals another approach to the meaning of the associativity condition. We define an operator $T \to T^*$ and are primarily interested in the question, whether or not T^* is associative. We conjecture, but have not yet proved, that, given that T is a t-norm, T^* is also a t-norm only if T is one of the basic t-norms T_w, T_m, Prod, Min, or an ordinal sum formed from these.

12a Definition. If $S \subseteqq [0, 1]$, we define $1-S$ to be the set

$$\{1-x : x \in S\}.$$

12b Definition. Let (S, σ) be a semigroup with $S \subseteqq [0, 1]$. The *invert* of (S, σ) is the set $1-S$ under the binary operation σ^* given by

$$x\sigma^* y = \max \begin{cases} x+y-1+(1-x)\sigma(1-y) \\ 0. \end{cases}$$

We remark that the invert T^* of a t-norm T may be conceived geometrically as follows: (1) Imagine the graph of T drawn over the unit square; (2) Rotate the square and graph 180° about the center point $(\frac{1}{2}, \frac{1}{2})$— a point (x, y) now bears the value $T(1-y, 1-x)$, which equals $T(1-x, 1-y)$ by the symmetry of T; (3) Restore the correct boundary values for a t-norm by adding $x+y-1$ to $T(1-x, 1-y)$; (4) At each point (x, y) where the value of $x+y-1+T(1-x, 1-y)$ is negative, replace this value by zero.

12c Definition. We shall write $T \in \mathcal{M}$ (for "of moderate growth") if for all $0 \leqq y \leqq \leqq w \leqq 1$ and $0 \leqq z \leqq x \leqq 1$,

$$T(w, x) - T(y-z) \leqq w-y+x-z. \tag{2}$$

The following remarks are easily verified:

(i) If $T \in \mathcal{M}$, then T is continuous.

(ii) If $T \in \mathcal{M}$ and $T_2'(x)$ exists on $(0, 1)$, then $T_2'(x) \leqq 2$ for all $x \in (0, 1)$, since $T_2(x) - T_2(y) \leqq 2(x-y)$.

(iii) Inequality (2) can be written as

$$(S-T)(w, x) \geqq (S-T)(y, z),$$

i.e., the two-place function "sum minus T" is non-decreasing in each place over the unit square.

12 Theorem. *Let T be a t-norm. Then* (i) *the t-norm boundary and symmetry conditions are satisfied by T^*, and* (ii) *if $T \in \mathcal{M}$ the t-norm monotonicity condition is satisfied by T^*. As a partial converse, if $T(x, y) \geqq x+y-1$ for all $(x, y) \in I^2$, and if T^* satisfies the monotonicity conditions, then $T \in \mathcal{M}$.*

PROOF. (i) Clearly, the symmetry of T implies that of T^*. As for the boundary conditions, $T^*(0, y)$ is clearly 0, as is $T^*(x, 0)$, and $T^*(1, y)$ is clearly y, and $T^*(x, 1)$ is x.

(ii) Suppose $T \in \mathscr{M}$ but that for some a, b, c, d with $0 \leqq a \leqq c \leqq 1$ and $0 \leqq b \leqq d \leqq 1$, we have

$$T^*(a, b) > T^*(c, d).$$

Then

$$a+b-1+T(1-a, 1-b) > c+d-1+T(1-c, 1-d),$$

so

$$T(1-a, 1-b)-T(1-c, 1-d) > c-a+d-b. \tag{3}$$

But $0 \leqq 1-c \leqq 1-a \leqq 1$ and $0 \leqq 1-d \leqq 1-b \leqq 1$, so that, since $T \in \mathscr{M}$,

$$T(1-a, 1-b)-T(1-c, 1-d) \leqq (1-a)-(1-c)+(1-b)-(1-d) = c-a+d-b,$$

contrary to (3). Therefore, if $T \in \mathscr{M}$, then T^* satisfies the t-norm monotonicity condition.

Conversely, suppose $T^*(a, b) \leqq T^*(c, d)$ for all a, b, c, d with $0 \leqq a \leqq c \leqq 1$ and $0 \leqq b \leqq d \leqq 1$. Then

$$\max\begin{cases} a+b-1+T(1-a, 1-b) \\ 0 \end{cases} \leqq \max\begin{cases} c+d-1+T(1-c, 1-d) \\ 0. \end{cases}$$

With $T(x, y) \geqq x+y-1$ for all $(x, y) \in I^2$, we have

$$T(1-a, 1-b) \geqq 1-a+1-b-1,$$

so that

$$a+b-1+T(1-a, 1-b) \geqq 0.$$

Similarly,

$$c+d-1+T(1-c, 1-d) \geqq 0;$$

thus,

$$a+b-1+T(1-a, 1-b) \leqq c+d-1+T(1-c, 1-d),$$

so

(4)

$$T(1-a, 1-b)-T(1-c, 1-d) \leqq c-a+d-b = (1-a)-(1-c)+(1-b)-(1-d).$$

Thus for any prescribed w, x, y, z with $0 \leqq y \leqq w \leqq 1$ and $0 \leqq z \leqq x \leqq 1$, we may set

$$a = 1-w, \quad b = 1-x, \quad c = 1-y, \quad d = 1-z,$$

and, since $0 \leqq a \leqq c \leqq 1$ and $0 \leqq b \leqq d \leqq 1$, inequality (4) becomes

$$T(w, x)-T(y, z) \leqq w-y+x-z.$$

13 Theorem. *Let (S, σ) be a semigroup with $S \subseteqq [0, 1)$. Then the invert of the invert of (S, σ) is (S, σ).*

PROOF.

$$\sigma^{**}(x,y) = \max\begin{cases} x+y-1+\sigma^*(1-x,1-y) \\ 0 \end{cases} =$$

$$= \max\begin{cases} x+y-1+\max\begin{cases} 1-x+1-y-1+\sigma(x,y) \\ 0 \end{cases} = \\ 0 \end{cases}$$

$$= \max\begin{cases} x+y-1+\max\begin{cases} 1-x-y+\sigma(x,y) \\ 0 \end{cases} = \\ 0 \end{cases}$$

$$= \max\begin{cases} \max\begin{cases} (x,y) \\ 0 \end{cases} = \\ 0 \end{cases}$$

$$= \sigma(x,y)$$

14 Examples. $\mathrm{Min}^* = \mathrm{Min}$, $\mathrm{Prod}^* = \mathrm{Prod}$, $T_m^* = T_w$ (hence $T_w^* = T_m$), and Ω^* (as in Theorem 10) is not associative.

These examples are easily checked.

15 Definition. Let (S, σ) be a semigroup with $S \subseteq [0, 1]$. Then (S, σ) is *invertible* if its invert is a semigroup. Suppose $\sigma(\cdot, \cdot)$ is non-decreasing in each place; then (S, σ) is ***t-invertible*** if it is invertible and $\sigma^*(\cdot, \cdot)$ is non-decreasing in each place. Thus a t-norm T is t-invertible if T^* is a t-norm (i.e., if $T \in \mathcal{M}$ and T^* is associative).

16 Theorem. *Let $T \in \mathcal{M}$ be a t-norm which is an ordinal sum of t-invertible semigroups (S_a, T_a), $a \in A$. Then T is t-invertible, and T^* is the corresponding ordinal sum of semigroups $(1-S_a, T_a^*)$, $a \in A$.*

PROOF.

$$T(x,y) = \begin{cases} T_a(x,y) & \text{if } (x,y) \in S_a \times S_a \text{ for some } a \in A \\ \min\{x,y\} & \text{otherwise} \end{cases}$$

$$T^*(x,y) = \max\begin{cases} x+y-1+T(1-x,1-y) \\ 0 \end{cases} =$$

$$= \max\begin{cases} x+y-1+\begin{cases} T_a(1-x,1-y) & \text{if } (1-x,1-y) \in S_a \times S_a \\ \min\{1-x,1-y\} & \text{otherwise} \end{cases} = \\ 0 \end{cases}$$

$$= \max\begin{cases} \begin{cases} x+y-1+T_a(1-x,1-y) & \text{if } (x,y) \in (1-S_a) \times (1-S_a) \\ \min\{x,y\} & \text{otherwise} \end{cases} = \\ 0 \end{cases}$$

$$= \begin{cases} \max\begin{cases} x+y-1+T_a(1-x,1-y) \\ 0 \end{cases} & \text{if } (x,y) \in (1-S_a) \times (1-S_a) \\ \min\{x,y\} & \text{otherwise} \end{cases} =$$

$$= \begin{cases} T_a^*(x,y) & \text{if } (x,y) \in (1-S_a) \times (1-S_a) \\ \min\{x,y\} & \text{otherwise.} \end{cases}$$

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