On dual semigroups

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Dual rings have been studied extensively by various authors, namely, BAER [1], KAPLANSKY [5], and NAKAYAMA [6] etc. The introductory notions of these papers use only the multiplicative properties of the elements of the ring. It seems therefore natural to ask how their results can be transferred to the theory of semigroups. This motivated SCHWARZ to introduce the notion of dual semigroups in [8].

This paper is a further study in obtaining structure theorems for dual semigroups satisfying maximum condition and 0-simple dual semigroups. Let us define that a semisimple-like semigroup is a semigroup with identity in which every right ideal is principally generated by an idempotent. Unlike in ring theory, a semisimple-like semigroup need not be a dual semigroup.

Throughout this paper every semigroup has 0 and contains more than one element. If A is a subset of a semigroup S, then we denote $A^R=\{x\in S\,|\,Ax=0\}$ and $A^L=\{x\in S\,|\,xA=0\}$. A semigroup S is said to be a dual semigroup if $B^{RL}=B$ for every left ideal B in S and $A^{LR}=A$ for every right ideal A in S. One-sided and two-sided ideals in a semigroup S are said to be proper if they are not either (0) or S. An element x in a semigroup S is a left or a right unit according as $tx=1$ or $xt=1$ for some $t\in S$. An element which is a left as well as a right unit is called a unit.

1. Semigroups with identity

In this section we shall mention the ideal structure of semigroups with identity. We note that a semigroup S which is not right simple has a unique maximal right ideal if S is principally generated, i.e., $S=f\cup fS$, for some $f\in S$.

1. 1. Theorem. *Let S be a semigroup with identity. Suppose S is not right simple. Then S has a unique maximal right ideal M such that either $SM=S$ (i.e., M is not a left ideal) or M is a left ideal. In the latter case M is a maximal left ideal; M is a maximal two-sided ideal and the set of non-units form an ideal.*

PROOF. If S is not right simple, S has a proper right ideal and so the set-theoretical union of all proper right ideals is the unique maximal right ideal M. If $SM\neq S$, then SM is a right ideal and hence $SM\subseteq M$. Thus M is a left ideal. Suppose M is not a maximal left ideal. Then there exists a proper left ideal L such that $M\subseteq L$.

Now $x \in L$, $x \notin M \Rightarrow xt=1$ for some $t \in S$. If $tx \neq 1$, then $t \in M$. So $xt \in M$ since M is a left ideal. Hence $1 \in M$, which is a contradiction. Thus $tx=1 \in L$, which is again a contradiction.

The last part follows from the observation that, if $x \notin M$, then x is a two-sided unit.

1.2 Corollary. *If S is a right cancellative semigroup with identity, then M, mentioned in* 1.1, *is a two-sided ideal and hence satisfies the second part of the theorem.*

PROOF. Let $r \in S$ and $m \in M$. If $rm \notin M$, then $rmS=S$ and so $(rm)t=1$. This implies $(mt)r(mt)=mt$ and $(mt)r=1$, by right cancellative condition. Hence $1 \in M$ since $m \in M$ and M is a right ideal. Thus we arrive at a contradiction.

1.3 *Remark.* Theorem 1.1 is not true if we assume that S has one-sided identity. Let S be the set of all non-zero complex numbers. Define an operation by $a\,0b=a\,|b|$. $(S, 0)$ is a semigroup. This has many maximal right ideals, in particular, $S \setminus R$ and $S \setminus (-R)$, where R is the set of all positive real numbers.

1.4 *Remark.* Even if $S=f \cup fS$, then S is either right simple or S has a unique maximal right ideal, which is the set-theoretical union of all right ideals not containing f.

Notation. If a semigroup S is not right simple, then we denote by (S, M) the semigroup with the unique maximal right ideal M.

1.5 *Definition.* The *Schwarz* radical of a semigroup with 0, is the set-theoretic union of all nilpotent right ideals, which coincides with the set-theoretic union of all nilpotent left ideals or the union of all nilpotent two-sided ideals.

1.6 Proposition. If (S, M) is a semigroup with identity, then $S=G \cup M$, where G is the semigroup of right units in case $S=SM$, while G is the group of units if $S \neq SM$. Furthermore if S has 0 and the radical of S is 0, then either $S=G \cup 0$, where G is the group of units or $M^L=0$.

PROOF. The first part is an easy consequence of 1.1. If $M=0$, then it can easily be verified that every $x \in G$ is a two-sided unit. If $M \neq 0$ and $M^L \neq 0$, then $M^L \subseteqq M$. For, if $x \in M^L$ and $x \notin M$, then $xt=1$. Suppose $t \in M$. Then $M=xtM \subseteqq xM=0$. If $t \notin M$, then $tm=1$, which implies that $xtm=x$ and hence $m=x$. Thus $tx=1 \in M^L$ since M^L is a left ideal and so $M=0$, which is a contradiction. Now $M^L \subseteqq M$ and so we have $(M^L)^2=0$, contradicting the fact that the radical is zero.

2. Dual semigroups with identity

All examples of dual semigroups, given by SCHWARZ [8], are semigroups without identity. An example of a dual semigroup with identity is $\{0, 1, a, a^2=0\}$. Clearly this is not a semisimple-like semigroup since the (right) ideal $\{0, a\}$ is not generated by an idempotent. In ring theory, 'the semisimplicity' implies for rings with minimum condition on right (or left) ideals the "dual" property. However, in semigroup theory, these two concepts are unrelated. For example, let $S=\{0, 1, a; a^2=a\}$. S is not a dual semigroup, but is a semisimple-like semigroup.

2. 1 Proposition. If M is a proper maximal right ideal in a dual semigroup S then $M^L \neq 0$ and M^L is a minimal left ideal. If M is the unique proper maximal right ideal, then M^L is the unique minimal left ideal and furthermore if $SM \neq S$, then $M^R = M^L$ and M^R is the only minimal left, minimal right and minimal two-sided ideal.

PROOF. If $M^L = 0$, then $M = M^{LR} = (0)^R = S$, a contradiction. Let A be a left ideal such that $A \subseteqq M^L$. Since $A^R \neq 0$ as above and $A^R \neq S$ by a similar argument, $A \subseteqq M^L \Rightarrow A^R \supseteqq M^{LR} = M \Rightarrow A^R = M \Rightarrow A = A^{RL} = M^L$. If M is the unique proper maximal ideal and if A is a minimal left ideal, then A^R is a maximal right ideal [8; lemma 1. 5] and so $A^R = M$. Hence $A = A^{RL} = M^L$. If $SM \neq S$, by 1. 1 M is a two-sided ideal. Then, as before it can be proved that M^R is a minimal right ideal. Since M is an ideal, evidently M^R and M^L are two-sided ideals and also minimal ideals. Hence $M^R = M^L$.

We shall now show that dual semigroups are divisible just like dual rings.

2. 2 Proposition. If (S, M) is a dual semigroup with an identity, then S is divisible, i.e., every cancellable element is an unit.

PROOF. Let x be a cancellable element. Clearly $x^R = 0$. So $Sx = (Sx)^{RL} = (0)^L = S$. Hence $1 = tx$ for some $t \in S$. This implies $x = xtx$ and hence $1 = xt$ by cancellative property of x.

2. 3 *Definition.* A semigroup S is said to be right uniform if the intersection of any two non-zero right ideals is non-zero.

We observe that the dual semigroups containing an identity belong to the class of right uniform dual semigroups but dual semigroups not containing an identity need not be right uniform. Consider the semigroup S [8; Example 5] with the multiplication table:

	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	c	0
b	0	0	b	0	d
c	0	0	c	0	a
d	0	d	0	b	0

S is dual but S is not right uniform sincs $aS \cap bS = 0$.

2. 4 Proposition. If S is a right uniform dual semigroup, then S has only one non-zero idempotent, which is the identity of S.

PROOF. Let e be a non-zero idempotent in S. If $x \in eS \cap e^R$, then $x = ex$ and $ex = 0$. Hence $x = 0$. This implies $e^R = 0$ by the right uniform property of S and so $(Se)^{RL} = (e^R)^L = (0)^L = S$. Then for every $x \in S$, $x = xe$. Thus e is a right identity of S. This implies that e is a two-sided identity by a lemma of Schwarz [8; 7. 1].

2. 5 Proposition. Let S be a dual semigroup with a non-zero idempotent. Then S is right uniform if and only if S has an identity.

PROOF. By virtue of 2. 4 it suffices to prove that S is right uniform if S has an identity. Let A and B be any two non-zero right ideals of S such that $A \cap B = 0$. Then $A^L \cup B^L = S$ by Corollary 1. 3 of [8]. This implies $1 \in A^L$ or B^L, i.e., $A=0$ or $B=0$, which is a contradiction.

By noting that the regular semigroups with 0 containing more than one element, have non-zero idempotents, we have by virtue of 2. 5,

2. 6 Corollary. Let S be a regular dual semigroup containing more than one element. If S is right uniform or equivalently S has an identity, then S is a group with zero.

Another consequence of Proposition 2. 5 is

2. 7 Theorem. *Let (S, M) be a dual semigroup with zero and with an identity. Then S is a semisimple-like semigroup iff S is a group with* 0.

Theorem 2. 7 can also be obtained as a consequence of the following, by noting that a semisimple-like semigroup has zero radical.

2. 8 Theorem. (SCHWARZ [8]. *Let (S, M) be a dual semigroup with* 0 *and with an identity. If the radical of S is* 0, *then $S = G \cup 0$.*

PROOF. By virtue of 1. 6, it suffices to show that $M^L \neq 0$. Suppose $M^L = 0$. Then $M = M^{LR} = (0)^R = S$, which is a contradiction.

2. 9 *Remark.* If we adjoin identity to a dual semigroup without identity, then the resulting semigroup need not be a dual semigroup. Let $S = \{0, a, b\}$ subject to the conditions $ab = ba = 0$, $a^2 = a$ and $b^2 = b$. S is a dual semigroup, but $S^* = \{0, 1, a, b\}$ is not a dual semigroup since $\{0, a, b\}^{RL} = (0)^L = S^*$.

3. 0-simple dual semigroups

Recently SCHWARZ [9] has proved that 0-simple dual semigroups are completely 0-simple. In view of this and by a result of REES [3; 83] and Proposition 2. 4 we have

3. 1 Theorem. *Let S be a* 0*-simple dual semigroup. Then S is a group with zero if either one of the conditions is satisfied:*

1. S has an identity,

2. S is right uniform.

4. Right Noetherian dual semigroups

4. 1 *Definitions.* A semigroup S is said to be right Noetherian if S satisfies maximal condition on right ideals. A semigroup S containing 0 is right uniform if any two non-zero right ideals have non-zero intersection.

4. 2 Lemma *Let S be a right Noetherian and right uniform dual semigroup. Then for any $a \in S$, either $Sa = S$ or a is nilpotent.*

PROOF. $a^R=0 \Rightarrow (Sa)^R=0$ since $a\in Sa$ in dual semigroups [8; Lemma 1. 6]. Therefore $Sa=(Sa)^{RL}=(0)^L=S$. Let $a^R\neq 0$ and a be not nilpotent. Then the ascending chain of right ideals $a^R\subseteqq(a^2)^R\subseteqq\cdots$ terminates by right Noetherian condition. So there exists a positive number k such that $(a^k)^R=(a^{k+1})^R=\cdots$. If $x\in(Sa^k)^R\cap a^kS$, then $x=a^ky$ and $a^{2k}x=0$. Hence $a^{3k}y=0$ and so $y\in(a^{3k})^R=(a^k)^R$. This implies $a^ky=0=x$. Thus $(Sa^k)^R\cap a^kS=0$. Since S is right uniform and a is not nilpotent by assumption, we must have $(Sa^k)^R=0$. Hence $Sa^k=(Sa^k)^{RL}=(0)^L=S$ and thus $Sa=Sa^k=S$.

4. 3 Theorem. Let S be right Noetherian and right uniform dual semigroup. Then S has a nilpotent radical N such that S/N (Rees factor semigroup of S modulo N) is a left 0-simple and left regular semigroup.

PROOF. Let $A=\{a\in S\,|\,a^R\neq 0\}$. Then A is a left ideal and is nil by 4. 2. Using the methods as in ring theory, it can be verified easily that A is a nilpotent left ideal and it can also be shown that the radical N of S is a nilpotent two-sided ideal. Then it follows that $A\subseteqq N\subseteqq A$ and thus $A=N$.

If $x\notin A$, then $x^2\notin A$ by 4. 2. Again applying 4. 2, we have $Sx=Sx^2=S$. By lemma 1. 6 of [8; 204], $x\in Sx$. Hence $x=tx^2$. The rest of the proof is obvious.

4. 4 Corollary. Let S be a right Noetherian and right uniform dual semigroup with zero radical. Then S is a left 0-simple and left regular semigroup.

4. 5 Corollary. If a p.r.i. semigroup S (every right ideal of S is of the form $f\cup fs$ for some $f\in S$) is a dual semigroup, then S has a radical N such that S/N is a left 0-simple and left regular semigroup.

PROOF. Since p.r.i. condition implies right Noetherian condition, it suffices to show that S is right uniform by virtue of 4. 3. If A and B are any two right ideals, then $A\cup B=e\cup eS$, $e\in S$. Hence $e\in A$ or $e\in B$. Thus $A\subseteqq B$ or $B\subseteqq A$, which implies that S is right uniform.

4. 6 *Remark.* If we just drop the right uniform condition, then 4. 3 need not be true. Consider the example of the semigroup S cited in 2. 3 S is not right uniform since $aS\cap bS=0$. The radical $N=0$. S is not a left 0-simple semigroup and also not a left regular semigroup.

One may naturally ask when S/N in 4. 3 is a group with zero. The answer is affirmative when S is commutative and also in the case when S contains the identity, which is shown in 4. 7. But the general case is still an open problem.

4. 7 Theorem. *Let S be a right Noetherian dual semigroup containing identity. Then $S=G\cup N$ and $G\cap N$ is empty, where G is the group of units of S and N is the Schwarz radical of S.*

PROOF. By 2. 5, S is right uniform. From 4. 3, it follows that, if $A=\{a\in S\,|\,a^R\neq 0\}$ then $A=N$. The proof is now completed by showing every $x\notin A$ is an unit. Clearly by 4. 2, $Sx=S$, i.e., there exists an element a such that $ax=1$. Then xa is an idempotent and so $xa=0$ or $xa=1$ since S has only two idempotents 0 and 1 by 2. 4. Since $xa=0$ implies $x=xax=0$, the proof is completed.

5. Ideal lattice of a semigroup

5. 1 *Definition*. A non-zero right ideal in a semigroup S with 0 is said to be not large, if there exists a non-zero right ideal B in S such that $A \cap B = 0$.

5. 2 Theorem. *Let S be a semigroup with* 0 *and without identity. Suppose S has more than two elements. Then the following conditions are equivalent*

i) *The right ideal lattice of S is a Boolean Algebra*

ii) *Every proper right ideal is not large*

iii) *S is the direct union of minimal right ideals.*

PROOF. By virtue of lemma 2 [2; 137] we have to show only that (ii)⇒(i). Assume (ii). If A is a right ideal, then there exists a non-zero right ideal B_1 such that $A \cap B_1 = 0$. Then by Zorn's lemma, we have $A \cap B = 0$, where $C \supset B$ and $A \cap C = 0$ implies $C = B$. The proof is completed by showing that $A \cup B$ is large, which implies evidently that $A \cup B = S$. Suppose $A \cup B$ is not large, then $(A \cup B) \cap C = 0$, for some right ideal C. Since $B \cup C$ contains C properly, $A \cap (B \cup C) \neq 0$ by the maximal property of B. Then $A \cap C = (A \cap B) \cup (A \cap C) = A \cap (B \cup C) \neq 0$, which implies that $(A \cup B) \cap C \neq 0$, which is a contradiction.

The right ideal lattice of a dual semigroup need not be a Boolean Algebra as seen in the following example:

	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	b	a
b	0	0	0	a	b
c	0	b	a	d	c
d	0	a	b	c	d

Conversely a semigroup with complemented right ideal lattice need not be a dua semigroup, which can be noted in the example of a semigroup $S = \{0, a, b\}$ where $a^2 = a$, $b^2 = 0 = ab = ba$, since $(0, a)^{RL} = (0, b)^L = (0, a, b) \neq (0, a)$. We are unable to prove that a dual semigroup with zero nilpotent radical has its right ideal lattice complemented.

5. 3 Proposition. Let S be a dual semigroup with 0 and with a complemented right ideal lattice. If its radical N is nilpotent, then $N = 0$.

PROOF. Suppose $N \neq 0$. Since the right ideal lattice is complemented, by condition (ii) of 5. 2, there exists a non-zero right ideal A such that $N \cap A = 0$. Then by lemma 1. 3 of [8; 203], $N^R \cup A^R = S$. But by lemma 5. 2 of [8; 220], every minimal right ideal is contained in N^R. This implies $S = N^R$ by condition (iii) of 5. 2. Hence $N = N^{RL} = S^L = 0$.

5. 4 Theorem. *Let S be a commutative dual semigroup with* 0 *in which every ideal is contained in a proper maximal ideal and the radical N is nilpotent. Assume S has at least two distinct maximal ideals. Then the ideal lattice of S is a Boolean Algebra if and only if the radical N of S is* 0.

PROOF. By virtue of 5. 3 it suffices to show that if the radical is 0, then the ideal lattice is complemented. Since $N=0$, $M\cap M^R = 0$ for every maximal ideal M. Also $M^R\neq 0$ by dual property. Then for any arbitrary ideal A in S, there exists a non-zero ideal B such that $A\cap B = 0$, since every ideal is included in a maximal ideal. Thus the result follows from 5. 2.

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(Received March 29, 1971.)