Height-slope and splitting length of abelian groups

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Introduction. The splitting problem in abelian groups asks when is an abelian group the direct sum of a torsion and a torsion free group. In [2] IRWIN, KHABBAZ, and RAYNA investigated the splitting problem via the concept of splitting length. The splitting length of an abelian group A, written $l(A)$, is the least positive integer n (otherwise infinity) for which $A \otimes \cdots \otimes A$, n factors, splits.

In seeking information on the splitting length of abelian groups, the major emphasis thus far has been to place conditions on the groups to force splitting lengths of one or infinity. In COROLLARY 2. 5 of [2], it was shown that if A is a group with $T(A)$ p-primary and $A/T(A)$ is not p-divisible, then either A splits or $A \otimes \cdots \otimes A$ will not split for any number of factors. A slightly different criterion was obtained by the authors in COROLLARY 2. 13 of [4]. Let A be a group. If there is an $a \in A$ with $h_p(a+T(A)) = 0$ and $A/T(A)/\langle a+T(A)\rangle$ is p-divisible for all primes p where the p-primary component of $T(A)$ is nonzero, then the splitting length of A is one or infinity.

In this paper we introduce the concept of "height-slope of a mixed group" — an internal invariant which specifies the splitting length of the group, whether finite or infinite. Height-slope is shown to be the proper setting for analysis of the class C_Λ of groups A having arbitrary Λ-primary maximal torsion subgroups $T(A)$ and arbitrary torsion free rank for which $A/T(A)$ is Λ-divisible. With the aid of this invariant, we are able to obtain generalizations of some of the results found in [2, 3].

The underlying concept for height-slope is that of a p-sequence. This was first introduced in [2] where it was shown to be related to the splitting of mixed groups A in C_p having torsion free rank one. In [4], p-sequences were extended to Λ-sequences where Λ is a nonempty set of primes. The authors prove the following result which will be needed in the sequel: let $A \in C_\Lambda$. Then A splits if and only if there is a maximal torsion free independent set M in A such that every $a \in M$ has a Λ-sequence.

In section 1 we present necessary notation and terminology. In section 2 we state necessary definitions and the MAIN THEOREM along with some lemmas (of

*) The initial ideas for this work are contained in the dissertation of the first named author written under the direction of Professor S. A. KHABBAZ at Lehigh University.

interest in their own right) which are needed for the proof. We also prove a theorem reducing the splitting problem of $A \in C_\Lambda$ to those subgroups of A which are p-divisible and of torsion free rank one. In the third section we give applications of the MAIN THEOREM.

1. Preliminaries

Throughout, "group" will mean "abelian group". For a group A, we denote its torsion subgroup by $T(A)$. For $n \in N$, $A^n = A \otimes \cdots \otimes A$, n factors, and for $a \in A$, $a^n = a \otimes \cdots \otimes a \in A^n$. The phrase "$A$ splits" will mean that $T(A)$ is a direct summand of A. Given a group A, a prime p, and $a \in A$, $h_p(a)$ will represent the p-height of a in A and $o(a)$ the order of a.

N will denote the positive integers with $N_0 = N \cup \{0\}$. Throughout, Λ will be an arbitrary nonempty subset of the set P of all primes.

$T(A)$ is Λ-primary if the p-primary component of $T(A)$ is nonzero for all $p \in \Lambda$ but is zero otherwise. A group A is Λ-divisible if $pA = A$ for all $p \in \Lambda$. For groups A_i, $i \in I$, $\sum_{i \in I} A_i$ denotes the direct sum.

2. Height-slope as an invariant for splitting length

Before we give our results we need some definitions.

Definitions. Let A be a group and $p \in P$.

(i) Let $a_0 \in A$. A sequence $\{a_i\} \subset A$ will be called a p-sequence for a_0 if $pa_1 = a_0$ and $pa_{n+1} = a_n$ for all $n \in N$ [2]. The element a_0 is said to have a Λ-sequence if a_0 has a p-sequence for every $p \in \Lambda$ [4]. Note that a_0 has a Λ-sequence if and only if a_0 belongs to the maximal Λ-divisible subgroup of A.

Suppose that $T(A)$ is Λ-primary.

(ii) Let $a \in A \setminus T(A)$, $\Lambda' \subset \Lambda$. The *height-slope of a with respect to* Λ', denoted $hs_{\Lambda'}(a)$, is defined by

$$\text{(a)} \quad hs_{\Lambda'}(a) = 0 \quad \text{if} \quad h_p(a) = 0 \quad \text{for infinitely many} \quad p \in \Lambda',$$

otherwise

$$\text{(b)} \quad hs_{\Lambda'}(a) = \sup_{n \in N} \inf_{p \in \Lambda'} \inf_{i \in N} \left\{ \frac{h_p(p^i na)}{i} \right\}.$$

(iii) For S a set of torsion free elements from A and $\Lambda' \subset \Lambda$, the *height-slope of S at* Λ', denoted $hs_{\Lambda'}(S)$, is defined by

$$hs_{\Lambda'}(S) = \inf_{a \in S} hs_{\Lambda'}(a).$$

For the case $S = \{a\}$ and $\Lambda' = \{p\}$ we use $hs_p(a)$.

(iv) The *height-slope of A*, denoted $hs(A)$, is now define by

$$hs(A) = \inf_{a \in A \setminus T(A)} hs_\Lambda(a) = hs_\Lambda(A \setminus T(A))$$

Finally,

(v) Let α be a real number, $p \in P$, and $a \in A$. We say that a satisfies *property* $(*)$ *for* α *and* p if there is a function $f: N_0 \to N_0$ nondecreasing to infinity such that $h_p(p^i a) > \alpha(i + f(i))$ for all $i \in N_0$.

Remarks. Observe that $hs(A)$ produces the "best" constant α with the property that for every torsion free $a \in A$ and every $\varepsilon > 0$, there exists an $n(a) \in N$ such that $h_p(p^i n(a)a) > (\alpha - \varepsilon)i$ for all $i \in N_0$ and $p \in \Lambda$.

Moreover, $hs(A)$ may be defined over any maximal torsion free independent set M in A, rather than all torsion free elements $a \in A$.

Also note that for any appropriate S and Λ', $hs_{\Lambda'}(S) \geqq hs(A)$.

We have laid the groundwork for

Main Theorem. *Let $A \in C_\Lambda$ have height-slope $\alpha \neq 0$. Then*

(a) *$l(A)=1$ if and only if for every torsion free $a \in A$, there is an $n(a) \in N$ such that $n(a)a$ belongs to the maximal Λ-divisible subgroup of A.*

(b) *$l(A)=2$ if and only if one of the following holds:*

(i) *$\alpha = \infty$ and A does not split, or*

(ii) *$\alpha \in (2, \infty)$, or*

(iii) *$\alpha = 2$ and for every torsion free $a \in A$ there is an $n(a) \in N$ such that for every $p \in \Lambda$, $n(a)a$ has property $(*)$ for 2 and p.*

(c) *$l(A)=n>2$ if and only if one of the following holds:*

(i) $\alpha \in \left(\frac{n}{n-1}, \frac{n-1}{n-2}\right]$, *or*

(ii) $\alpha = \frac{n}{n-1}$ *and for every torsion free $a \in A$ there is an $n(a) \in N$ such that for every $p \in \Lambda$, $n(a)a$ has property $(*)$ for α and p.*

Before we give any proofs we would like to give some examples to show the usefulness of height-slope. First we apply the Main Theorem to the examples given in [2] of groups with various splitting lengths.

Definition. For $\sigma = 2, 3, 4, \dots$ the group A_σ is the free abelian group generated by $a_0, a_1, a_2, \dots$ modulo the relators $p^{(\sigma-1)i}a_i - p^{(\sigma-2)i}a_0$.

Note that the height-slope of A_σ is $\frac{\sigma-1}{\sigma-2}$ for $\sigma > 2$ and infinite for $\sigma = 2$. Moreover, A_2 has no elements of infinite order which are contained in the maximal p-divisible subgroup, while for $\sigma > 2$, A_σ contains no torsion free elements satisfying property $(*)$ for $\frac{\sigma-1}{\sigma-2}$ and p. Thus applying the Main Theorem we have that for $\sigma = 2, 3, 4, \dots$, the group A_σ has splitting length σ.

This covers Corollaries 4. 2, 4. 3 and Theorem 4. 2 of [2].

Next we give a supply of examples of groups with infinite splitting length.

Example. Let A be defined on the generators $a_0, a_1, a_2, \dots$ subject to the relations $p^i a_0 = p^{i+f(i)} a_i$ where $f: N_0 \to N_0$ is a function nondecreasing to ∞ such that $\lim_{i \to \infty} (f(i)/i) = 0$. Observe then that $hs(A) = 1$ and so $l(A) = \infty$ (see Corollary 2. 7).

(Note that it is easy to construct a group $A \in C_\Lambda$ of infinite splitting length where $hs(A) = 0$.)

We begin the proof of the Main Theorem by rewriting, in a more general setting, some lemmas from [2]. The first two deal with choosing p-sequences in special forms and are needed in the proofs of the remaining lemmas.

Lemma 2. 1. *Let A and B be groups with $A/T(A)$ of rank one and $B/T(B)$ p-divisible. Suppose $a \otimes b \in A \otimes B$ is a torsion free element which has a p-sequence $\{c_i\}$. Then the c_i can be chosen such that $c_i = a \otimes b_i$, $b_i \in B$.*

PROOF. The proof is essentially the one given for LEMMA 3. 5 of [2].

One may remove the restriction on the rank of $A/T(A)$ by requiring instead that $A/T(A)$ be p-divisible.

Lemma 2. 2. *Let A and B be groups with $A/T(A)$ and $B/T(B)$ p-divisible. Let $a \otimes b \in A \otimes B$ be a torsion free element with a p-sequence $\{c_i\}$. Then the c_i can be chosen such that $c_i = a \otimes b_i$, $b_i \in B$.*

PROOF. Our proof is a modification of the one given for LEMMA 3. 7 of [2]. Embed $a+T(A)$ in a pure subgroup $S/T(A)$ of $A/T(A)$ which is of torsion free rank one. Note that S contains $T(A)$, is of torsion free rank one, A/S is p-divisible since $A/T(A)$ is, and S is pure in A. Moreover, A/S is torsion free: for suppose $na = s \in S$, $a \in A$. Then by the purity of $S/T(A)$ in $A/T(A)$, $n(a-s_1) \in T(A)$ for some $s_1 \in S$. Thus $a - s_1 \in T(A) \subset S$ and so $a \in S$.

Now by the exactness of $0 \to S \to A \to A/S \to 0$ and the torsion freeness of A/S we have $0 \to S \otimes B \to A \otimes B \to (A/S) \otimes B \to 0$ is exact. Since A/S is torsion free and p-divisible, $(A/S) \otimes B$ contains no p-torsion. This implies that the p-sequence $\{c_i\}$ for $a \otimes b$ in fact belongs to $S \otimes B$. LEMMA 2. 1 now applies to give the result.

Next we would like to prove the converses of LEMMAS 1 and 3 of [3] under fewer restrictions. These lemmas are concerned with the relationships between heights of elements $a_i \in A_i$, $1 \leqq i \leqq n$, and $a_1 \otimes \cdots \otimes a_n \in A_1 \otimes \cdots \otimes A_n$. They state sufficient conditions on the p-heights of $a_i \in A_i$ so that $a_1 \otimes \cdots \otimes a_n$ has infinite p-height or a p-sequence.

Lemma 2. 3. *Let A_j be groups with $a_j \in A_j$ torsion free elements, $1 \leqq j \leqq n$, and $p \in P$. If for every $1 \leqq j \leqq n$, $h_p(p^i a_j) > \frac{n}{n-1} i$ for all $i \in N_0$, then the p-height of $a_1 \otimes \cdots \otimes a_n$ in $A_1 \otimes \cdots \otimes A_n$ is infinite.*

PROOF. Since $h_p(p^i a_j) > \frac{n}{n-1} i$ for all $i \in N_0$, there exist $a_{j,k} \in A_j$ such that

$$p^{(n-1)k} a_j = p^{kn+1} a_{j,k} \quad \text{for} \quad 0 \leqq k \text{ and } 1 \leqq j \leqq n.$$

Now

$$p^{n(k+1)+1} a_{j,k+1} = p^{(n-1)(k+1)} a_j = p^{n-1}(p^{(n-1)k} a_j) = p^{(n-1)}(p^{kn+1} a_{j,k}) = \\ = p^{n(k+1)} a_{j,k}.$$

Therefore, $(\#)\, p^{n(k+1)} a_{j,k} = p^{n(k+1)+1} a_{j,k+1}$ holds for $k \in N_0$ and $1 \leqq j \leqq n$. But

$$a_1 \otimes \cdots \otimes a_n = (pa_{1,0}) \otimes \cdots \otimes (pa_{n,0}) = p^n (a_{1,0} \otimes \cdots \otimes a_{n,0}).$$

Now by iterating the use of $(\#)$ component-wise for the various k's we have

$$a_1 \otimes \cdots \otimes a_n = p^n (a_{1,0} \otimes \cdots \otimes a_{n,0}) = \\ = p^{2n} (a_{1,1} \otimes \cdots \otimes a_{n,1}) = \cdots = p^{(k+1)n} (a_{1,k} \otimes \cdots \otimes a_{n,k}) \quad \text{for all} \quad k \in N_0.$$

Hence our conclusion.

Lemma 2. 4. *Let A_j be groups with $a_j \in A_j$ of infinite order, $1 \leqq j \leqq n$, and $p \in P$. If for some $1 \leqq l \leqq n$, a_l has property $(*)$ for $\alpha = \frac{n}{n-1}$ and p, and for all other $1 \leqq j \leqq n$, $h_p(p^i a_j) > \frac{n}{n-1} i$ for all $i \in N_0$, then $a_1 \otimes \cdots \otimes a_n$ belongs to the maximal p-divisible subgroup of $A_1 \otimes \cdots \otimes A_n$ (i.e. has a p-sequence).*

Proof. We shall give a proof which covers the case $n>2$ since these can be handled with one method. The case $n=2$ requires a separate but similar proof which we shall omit. Let $f: N_0 \to N_0$ be a function for a_l under property $(*)$. Let n_i be a subsequence for which $f(n_i)$ is strictly increasing. Without loss of generality we may assume that f increases as slowly as possible, i.e. $f(n_{i+1}) = f(n_i)+1$ and that a_n is a_l. Throughout the proof we shall use the following convention: by the element $a_{j,i}$ we shall mean that $a_{j,i} \in A_j$ and $a_{j,i}$ is a p-root to $p^i a_j$, i.e., $p^i a_j = p^n a_{j,i}$ for some $n \in N$. Since the proof is rather tedious we shall exhibit c_1, the first term of the p-sequence, and indicate how to obtain c_n in general. The equalities that follow are consequences of the hypotheses on the p-heights of $a_1, \ldots, a_n$.
Now

$$a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1} \otimes a_n = p^{(n-1)}(a_{1,0} \otimes \cdots \otimes a_{n-1,0}) \otimes a_n =$$
$$= p^{(n+1)}(a_{1,0} \otimes a_{2,0} \otimes \cdots \otimes a_{n-1,0} \otimes a_{n,n-1}) =$$
$$= p^{(n+2)}(a_{1,n} \otimes a_{2,0} \otimes \cdots \otimes a_{n-1,0} \otimes a_{n,n-1}) =$$
$$= p^{(n+3)}(a_{1,n} \otimes a_{2,n} \otimes a_{3,0} \otimes \cdots \otimes a_{n-1,0} \otimes a_{n,n-1}) = \cdots$$
$$\cdots = p^{2n}(a_{1,n} \otimes a_{2,n} \otimes \cdots \otimes a_{n-1,n} \otimes a_{n,n-1}) =$$
$$= p^{(2n+1)}(a_{1,n} \otimes a_{2,n} \otimes \cdots \otimes a_{n-1,n} \otimes a_{n,2(n-1)}) =$$
$$= p^{(2n+2)}(a_{1,2(n-1)+1} \otimes a_{2,n} \otimes \cdots \otimes a_{n-1,n} \otimes a_{n,2(n-1)}) = \cdots$$
$$\cdots = p^{3n}(a_{1,2(n-1)+1} \otimes a_{2,2(n-1)+1} \otimes \cdots \otimes a_{n-1,2(n-1)+1} \otimes a_{n,2(n-1)}) = \cdots$$
$$\cdots = p^{(k+1)n}(a_{1,k(n-1)+1} \otimes \cdots \otimes a_{n-1,k(n-1)+1} \otimes a_{n,k(n-1)}).$$

Now suppose $k_1(n-1) < n_1 \leqq (k_1+1)(n-1)$. Hence,

$$h_p(p^{(k_1+1)n} a_{n,k_1(n-1)}) = h_p(p^{(k_1+1)(n-1)} a_n) >$$
$$\frac{n}{n-1}((k_1+1)(n-1)+f(n_1)) \geqq \frac{n}{n-1}((k_1+1)(n-1)+1) = (k_1+1)n+1+\frac{1}{n-1},$$

so that the p-height is at least $(k_1+1)n+2$. Thus

$$p^{(k_1+1)n}(a_{1,k_1(n-1)+1} \otimes \cdots \otimes a_{n-1,k_1(n-1)+1} \otimes a_{n,k_1(n-1)}) =$$
$$= p^{(k_1+1)n+2}(a_{1,k_1(n-1)+1} \otimes \cdots \otimes a_{n-1,k_1(n-1)+1} \otimes a_{n,(k_1+1)(n-1)}).$$

Let

$$c_1 = p^{(k_1+1)n+1}(a_{1,k_1(n-1)+1} \otimes \cdots \otimes a_{n-1,k_1(n-1)+1} \otimes a_{n,(k_1+1)(n-1)}).$$

Continuing the procedure with c_1 one has

$$c_1 = p^{(k_1+k_2+1)n}(a_{1,(k_1+k_2)(n-1)+1} \otimes \cdots \otimes a_{n-1,(k_1+k_2)(n-1)+1} \otimes a_{n,(k_1+k_2)(n-1)-1}).$$

The second jump in p-heights occurs when

$$(k_1+k_2)(n-1)-1 < n_2 \leqq (k_1+k_2+1)(n-1)-1.$$

Then one can define c_2. The above process remains intact due to the fact that f compensates for the loss in the power of p when a choice of a p-sequence element is made.

In the special case where $A_i=A$ for $1\leqq i\leqq n$ we have the following corollaries. Their proofs are easily derivable from LEMMAS 2. 2, 2. 3, and 2. 4 and LEMMAS 1 and 3 in [3].

COROLLARY 2. 5. Let A be a group, $a\in A$ with $o(a)=\infty$, and $p\in P$. Then there is a $k\in N$ such that $h_p(ka^n)=\infty$ in A^n if and only if $hs_p(a)>\frac{n}{n-1}$, or $hs_p(a)=\frac{n}{n-1}$ and for some $m\in N$, $h_p(p^i ma)>\frac{n}{n-1}i$ for all $i\in N_0$.

COROLLARY 2. 6. Let A be a group such that $A/T(A)$ is p-divisible and let $a\in A$ have infinite order. Then a^n belongs to the maximal p-divisible subgroup of A^n if and only if a satisfies property $(*)$ for $\frac{n}{n-1}$ and p.

We are now ready for

PROOF of the MAIN THEOREM.

(a) is stated for the sake of completeness and is THEOREM 3. 4 of [4].

(b) Suppose $l(A)=2$. Then $A\otimes A = T(A\otimes A)\oplus F$ for some torsion free subgroup F of $A\otimes A$. Let $a\in A$ have infinite order. Then $a\otimes a$ is torsion free so that for some $k\in N$, $k(a\otimes a)\in F$. Now, since $A/T(A)$ is Λ-divisible, F is Λ-divisible and hence $k(a\otimes a)$ has a Λ-sequence. By COROLLARY 2. 6, ka satisfies property $(*)$ for 2 and all $p\in\Lambda$. Since a was an arbitrary torsion free element of A, our conclusion holds for all such a.

It is clear that if condition (b) (iii) implies that $l(A)=2$, then so will (b) (i) or (ii). Now, LEMMA 2. 4 implies that every $n(a_1)a_1\otimes n(a_2)a_2$ has a Λ-sequence in A^2, where the a_i are torsion free and $n(a_i)\in N$. But if we take M to be a maximal torsion free independent set in A, the set of all $n(a_1)a_1\otimes n(a_2)a_2$, $a_i\in M$, forms a maximal torsion free independent set in A^2. Hence, by THEOREM 3. 4 of [4], A^2 splits, i.e., $l(A)\leqq 2$. If $l(A)=1$, then $hs(A)=\infty$, which is a contradiction to our hypothesis. This concludes (b). The proof of (c) is similar to (b) with obvious modifications.

An immediate consequence of the MAIN THEOREM is

COROLLARY 2. 7. Let $A\in C_\Lambda$. $l(A)=\infty$ if and only if $hs(A)=0$ or $hs(A)=1$.
Next we prove a theorem analogous to THEOREM 2 in [3].

Theorem 2. 8. *Let $A\in C_\Lambda$ and let $\pi: A\to A/T(A)$ denote the natural epimorphism. Then A splits if and only if $\pi^{-1}(G)$ splits for every torsion free p-divisible rank one subgroup G of $A/T(A)$ where $p\in\Lambda$.*

PROOF. The necessity being clear, we prove the sufficiency. Let $a\in A$ be torsion free. Embed $\pi(a)$ in a Λ-divisible subgroup G of $A/T(A)$ which is of torsion free rank one. By hypothesis, $\pi^{-1}(G) = T(A) \oplus F$ with $F \cong G$ torsion free Λ-divisible. Now $a \in \pi^{-1}(G)$ so that $n(a)a \in F$ for some $n(a) \in N$. Thus $n(a)a$ has a Λ-sequence in $F \subseteq A$. Hence, by the MAIN THEOREM, $l(A)=1$, i.e. A splits.

3. Applications and examples.

In this section we present some corollaries of the MAIN THEOREM and give some examples to help clarify the concept of height-slope.

COROLLARY 3. 1. Let $A \in C_\Lambda$ and M a maximal torsion free independent set in A. Then the following are equivalent:

(i) A^n splits,

(ii) For every $a \in M$, there is a $k \in N$ such that for every $p \in \Lambda$, ka satisfies property $(*)$ for $\frac{n}{n-1}$ and p, and

(iii) For any $a_i \in M$, $1 \leqq i \leqq n$, there is a $k \in N$ such that $k(a_1 \otimes \cdots \otimes a_n)$ has a Λ-sequence in A^n.

PROOF. This is an easy consequence of COROLLARY 2. 6 and THEOREM 3. 4 of [4].

Next we generalize THEOREM 1 of [3].

COROLLARY 3. 2. Suppose $X_i \in C_\Lambda$ with $l(X_i) \leqq n$ for $1 \leqq i \leqq n$. Then $X_1 \otimes \cdots \otimes X_n$ splits.

PROOF. By COROLLARY 3. 1, we can choose maximal torsion free independent sets $M_i \subseteq X_i$ such that every $x_i \in M_i$ satisfies property $(^*)$ for $\frac{n}{n-1}$ and each $p \in \Lambda$. Let $M = \{x_1 \otimes \cdots \otimes x_n : x_i \in M_i\}$. Now M is a maximal torsion free independent set in $X_1 \otimes \cdots \otimes X_n$ and by LEMMA 2. 4 every $x_1 \otimes \cdots \otimes x_n \in M$ has a Λ-sequence in $X_1 \otimes \cdots \otimes X_n$. Applying THEOREM 3. 4 of [4], we have that $X_1 \otimes \cdots \otimes X_n$ splits.

COROLLARY 3. 3. Let $X = \sum_{i \in I} X_i$ where each $X_i \in C_\Lambda$. Then $l(X) = \sup_{i \in I} l(X_i)$.

PROOF. Note first that $X^n = \sum_{i_1, \ldots, i_n \in I} (X_{i_1} \otimes \cdots \otimes X_{i_n})$. Since X^n splits if and only if every one of its summands splits, and X_i^n is a summand for every $i \in I$, $l(X) \geqq \sup_{i \in I} l(X_i)$. The reverse inequality follows from COROLLARY 3. 2.

The conclusion of LEMMA 5 of [3] can be proven for the class C_Λ under fewer hypotheses.

COROLLARY 3. 4. Let $A \in C_\Lambda$ and let B be a subgroup of A such that A/B is torsion free. Then $l(B) \leqq l(A)$.

PROOF. It is easy to show that $B \in C_\Lambda$. By the definition of height-slope and the subsequent REMARK, $hs(B) \geqq hs(A)$. Also, if $hs(B) = hs(A) = \frac{n}{n-1}$ and every torsion free element $a \in A$ has property $(*)$ for $\frac{n}{n-1}$ and each $p \in \Lambda$, then so will the torsion free elements $b \in B$. Thus, by the MAIN THEOREM ,$l(B) \leqq l(A)$.

By a similar comparison of the height-slopes of A and B we have

COROLLARY 3. 5. Let $A \in C_\Lambda$, $B \subseteq A$ with $B \in C_{\Lambda'}$. Then $l(B) \leqq l(A)$.
Analogous to LEMMAS 4. 2 and 4. 3 of [2] we have

COROLLARY 3. 6. Let $A \in C_\Lambda$, and $f: A \to B$ a homomorphism with $T(B)$ Λ'-primary for some $\Lambda' \subseteqq \Lambda$. If

(i) f is an epimorphism, or

(ii) $B \in C_{\Lambda'}$ and $f(A)$ contains a maximal torsion free independent set in B,

then $l(B) \leqq l(A)$.

PROOF. (i) It is straightforward to show that $B \in C_{\Lambda'}$. A direct comparison of the height-slopes of A and B gives us the result.

(ii) If $l(A) = \infty$, we are finished. Suppose $l(A) = n < \infty$. $f: A \to B$ induces a homomorphism $f^n: A^n \to B^n$. Let M be a maximal torsion free independent set contained in the image of f. Let $b_i = f(a_i) \in M$, $1 \leqq i \leqq n$. Since A^n splits and $A^n/T(A^n)$ is Λ-divisible, there exists a $k \in N$ such that $k(a_1 \otimes \cdots \otimes a_n)$ has a Λ-sequence. Thus, $k(b_1 \otimes \cdots \otimes b_n)$ has a Λ-sequence and by COROLLARY 3. 1, B^n splits.

COROLLARY 3. 7. Let $A, B \in C_\Lambda$. Then $l(A \otimes B) \leqq \min(l(A), l(B))$.

PROOF. The proof is identical to the one given for LEMMA 4. 1 of [2] where $A, B \in C_p$.

The measure of the length of the tensor product given in COROLLARY 3. 7 is insufficient if A and B do not belong to the same class C_Λ.

COROLLARY 3. 8. Let $A \in C_{\Lambda_1}$, $B \in C_{\Lambda_2}$. Then $l(A \otimes B) \leqq \max(l(A), l(B))$.

PROOF. $A \in C_{\Lambda_1}$, $B \in C_{\Lambda_2}$ implies that $A \otimes B \in C_\Lambda$ where $\Lambda \subseteqq \Lambda_1 \cup \Lambda_2$. The result follows upon comparison of the height-slope of $A \otimes B$ with that of A and B.

In COROLLARY 3. 8 it is possible for equality to hold. Consider the groups $A_{2,p}$, $A_{3,q}$ where $\sigma = 2$ and 3 in the definition of A_σ given earlier but for two distinct primes p, q, respectively. Suppose $l(A_{2,p} \otimes A_{3,q}) < 3$. Since $A_{2,p}^2 \otimes A_{3,q}^2 / T(A_{2,p}^2 \otimes A_{3,q}^2)$ is $\{p, q\}$-divisible, $k(a_{0,p}^2 \otimes a_{0,q}^2)$ has a q-sequence for some $k \in N$. By LEMMA 2. 1, the q-sequence can be chosen in the form $ka_{0,p}^2 \otimes x_i$, $x_i \in A_{3,q}^2$. However, since the q-height of $ka_{0,p}^2$ is finite and $x_i - qx_{i+1}$ are of unbounded orders, this is impossible unless x_i is a q-sequence for $ka_{0,q}^2$. But this would imply that $l(A_{3,q}) = 2$, a contradiction.

We close with two examples to show that extreme care must be taken when the height-slope is an end-point.

EXAMPLE 3. 9. Consider the group A defined on the generators $a_0, a_1, a_2, \ldots$ subject to the relations $p^i a_0 = p^{2i - [\sqrt{i}]} a_i$, where [ ] denotes the greatest integer function. It is easy to see that $hs(A) = 2$, yet A^2 contains no torsion free elements of infinite p-height. This clarifies the hypotheses of COROLLARY 2. 5.

Finally we give an example of a group A such that A^2 has an element of infinite p-height but none with a p-sequence. In fact this group will have the property that $h_p(p^i a_0) - 2i$ is unbounded yet property $(*)$ does not hold for 2 and p.

EXAMPLE 3. 10. Let $n_i = 10^i$. Let A be the group defined on the generators $a_0, a_1, a_2, \ldots$ subject to the relations $a_0 = p^{n_1} a_1$, $p^{n_i} a_0 = p^{n_i + n_{i+1}} a_{i+1}$ for $i \in N$. The height-slope of A is 2, $h_p(p^{n_i} a_0) - 2n_i$ is unbounded, yet $l(A) = 3$.

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