An algebraic approach to distributions on an open interval

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The space $\mathscr{D}$ of complex-valued infinitely differentiable functions on the reals having compact support is closed under convolution. If F is a distribution on the reals then the convolution operator defined by F maps $\mathscr{D}$ into the space of infinitely differentiable functions and commutes with convolution in $\mathscr{D}$. In [7] it is shown that every such operator is defined by some distribution. In the present article we show that the space of distributions on any open interval (a, b) is algebraically isomorphic to a space of operators which commute with convolution; this space is denoted by $P(a, b)$. We also show that the family of mappings in $P(-\infty, b)$ whose ranges consist of functions with left-bounded support correspond to the distributions on $(-\infty, b)$ which have left-bounded support.

Preliminary definitions

If f is any function defined on the reals we define the support of f, denoted $\operatorname{supp} f$, to be the closure of the set $\{t:f(t)\neq 0\}$. For any open subset Ω of the reals let $C(\Omega)$ be the set of all continuous complex-valued functions on Ω and $C^\infty(\Omega)$ be the set of all infinitely differentiable functions on Ω. For any compact subset K of Ω we denote by $\mathscr{D}(K)$ the space of all φ in $C^\infty(\Omega)$ such that $\operatorname{supp}\varphi\subset K$. The space $\mathscr{D}(\Omega)$ is the union of the spaces $\mathscr{D}(K)$ where K ranges over the compact subsets of Ω. The space $\mathscr{D}\big((-\infty, \infty)\big)$ will also be denoted simply by $\mathscr{D}$. The dual of $\mathscr{D}(\Omega)$, that is, the space of distributions on Ω, is denoted by $\mathscr{D}'(\Omega)$. If F belongs to $\mathscr{D}'(\Omega)$ and φ belongs to $\mathscr{D}(\Omega)$ the scalar which F assigns to φ will be written $\langle F(\tau), \varphi(\tau)\rangle$. If f is a locally integrable function on Ω we shall write $\partial^0 f$ for the element of $\mathscr{D}'(\Omega)$ defined by

$$\langle \partial^0 f(\tau), \varphi(\tau)\rangle = \int_{-\infty}^{\infty} f(u)\varphi(u)\,du \qquad (\text{all } \varphi\in\mathscr{D}(\Omega)).$$

Thus, $\partial^0 f$ is the regular distribution corresponding to the function f. The support of a distribution F in $\mathscr{D}'(\Omega)$ is defined to be the complement, with respect to Ω, of the largest open set on which F vanishes.

For any φ and ψ in $\mathscr{D}$ we may define the convolution $\varphi^*\psi$:

$$\varphi^*\psi(t) = \int_{-\infty}^{\infty} \varphi(u)\psi(t-u)\,du \qquad \text{(all real } t).$$

Then $\varphi^*\psi$ belongs to $\mathscr{D}$ with $(\varphi^*\psi)'=\varphi'^*\psi=\varphi^*\psi'$ (see sections 5. 4 and 5. 5 in [9]).

We shall use the following convention: If x is any real number then $x+\infty = \infty+x = \infty$ and $x-\infty = -\infty+x = -\infty$; also, $\infty-(-\infty) = \infty$ and $\varepsilon\infty=\infty$ for all $\varepsilon>0$.

The space $P(a, b)$

We assume $-\infty\leqq a<b\leqq\infty$ and $r = \frac{1}{2}(b-a)$. Thus, if $a=-\infty$ or $b=\infty$ then $r=\infty$. If $0\leqq x<r$ we write $U_x = (a+x, b-x)$; thus, U_0 is the interval (a, b). For each φ in $\mathscr{D}((-r, r))$ there exists a smallest nonnegative number $|\varphi|$ such that $\operatorname{supp}\varphi\subset\subset[-|\varphi|, |\varphi|]$.

Suppose $0\leqq c<r$ and let f belong to $C(U_c)$. For each φ in $\mathscr{D}((-r, r))$ such that $c+|\varphi| < r$ the function $f^*\varphi$ defined on $U_{c+|\varphi|}$ by

$$f^*\varphi(t) = \int_{t-|\varphi|}^{t+|\varphi|} f(u)\varphi(t-u)\,du$$

belongs to $C^\infty(U_{c+|\varphi|})$ with $(f^*\varphi)'=f^*\varphi'$ (see [3, Theorem 250]). Thus, if $f\in C((a, b))$ then the function $f^*\varphi\in C^\infty(U_{|\varphi|})$ for all $\varphi\in\mathscr{D}((-r,r))$; we denote by $\{f\}$ the mapping that assigns to each φ in $\mathscr{D}((-r, r))$ the function $f^*\varphi$ in $C^\infty(U_{|\varphi|})$.

We observe that if φ and ψ belong to $\mathscr{D}((-r, r))$ with $|\varphi|+|\psi| < r$ then $\varphi^*\psi$ belongs to $\mathscr{D}((-r, r))$ with $|\varphi^*\psi| \leqq |\varphi|+|\psi|$ (see [8], Prop. 26. 7),

Definition. Let $P(a, b)$ denote the family of all the mappings T which assign to each φ in $\mathscr{D}((-r, r))$ a function $T\varphi$ in $C^\infty(U_{|\varphi|})$ and which satisfy the equation

$$T(\varphi^*\psi) = (T\varphi)^*\psi$$

on $U_{|\varphi|+|\psi|}$ for all φ and ψ in $\mathscr{D}((-r, r))$ with $|\varphi|+|\psi| < r$.

It is a consequence of [3], Theorem 281 that $\{f\}$ belongs to $P(a, b)$ for all f in $C((a, b))$. As we shall presently see, every F in $\mathscr{D}((a, b))$ defines an element $F\to\{F\}$ of $P(a, b)$.

Definition. Suppose F belongs to $\mathscr{D}'((a, b))$ and φ belongs to $\mathscr{D}((-r, r))$. We denote by $\{F\}\varphi$ the function that assigns to any t in $U_{|\varphi|}$ the number $\langle F(\tau), \varphi(t-\tau)\rangle$; consequently,

$$\{F\}\varphi(t) = \langle F(\tau), \varphi(t-\tau)\rangle \qquad (\text{all } t\in U_{|\varphi|}).$$

Further, let $\{F\}$ be the mapping $\varphi\to\{F\}\varphi$.

Theorem 1. *For each F in $\mathscr{D}'((a, b))$ the mapping $\{F\}$ belongs to $P(a, b)$.*

The theorem follows from generalizations of Theorem 5. 5-1, Theorem 5.4-3 and Corollary 5. 4-1a in [9]; the proof is omitted. It is easily seen that $F \to \{F\}$ is a linear mapping (of $\mathscr{D}'((a, b))$ into $P(a, b)$). We shall now show that the two spaces are, in fact, algebraically isomorphic.

For each ψ in $\mathscr{D}$ and any real t we denote by ψ_t the element of $\mathscr{D}$ defined by $\psi_t(\tau) = \psi(t-\tau)$.

Lemma. *For each $\psi \in \mathscr{D}((a, b))$ there eixists t such that $\psi_t \in \mathscr{D}((-r, r))$ and $t \in U_{|\psi_t|}$.*

Proof. If $\psi \in \mathscr{D}((a, b))$ there exist α and β such that $\operatorname{supp}\psi \subset [\alpha, \beta] \subset (a, b)$. If we define $t = (1/2)(\alpha+\beta)$ then $\psi_t \in \mathscr{D}((-r, r))$ and $t \in U_{|\psi_t|}$.

Theorem 2. *The mapping $F \to \{F\}$ is a linear bijection of $\mathscr{D}'((a, b))$ onto $P(a, b)$.*

Proof. We show first that the mapping is "onto". Let $T \in P(a, b)$ be given. Let δ_n $(n=1, 2, \ldots)$ be a "δ-sequence" in $\mathscr{D}((-r, r))$. Then

$$T\varphi(t) = \lim_{n\to\infty} T\varphi^* \delta_n(t) = \lim_{n\to\infty} T\delta_n^* \varphi(t) \tag{1}$$

for all φ in $\mathscr{D}((-r, r))$ and all t in $U_{|\varphi|}$. Define

$$f_n(t) = \begin{cases} T\delta_n(t) & \text{for } a+\dfrac{2}{n} < t < b-\dfrac{2}{n} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Then f_n is a locally integrable function on (a, b) since $T\delta_n \in C(U_{1/n})$. For any ψ in $\mathscr{D}((a, b))$ we may find t such that $\psi_t \in \mathscr{D}((-r, r))$ and $t \in U_{|\psi_t|}$. Choosing n sufficiently large so that

$$a+\frac{2}{n} < t-|\psi_t| \leqq t+|\psi_t| < b-\frac{2}{n}$$

we have $f_n(u) = T\delta_n(u)$ for $t-|\psi_t| \leqq u \leqq t+|\psi_t|$; therefore,

$$\langle \partial^0 f_n(\tau), \psi(\tau)\rangle = \int_a^b f_n(u)\psi(u)\,du = \int_a^b f_n(u)\psi_t(t-u)\,du =$$

$$= \int_{t-|\psi_t|}^{t+|\psi_t|} f_n(u)\psi_t(t-u)\,du = \int_{t-|\psi_t|}^{t+|\psi_t|} T\delta_n(u)\psi_t(t-u)\,du.$$

We may now use (1) to conclude that $T\psi_t(t) = \lim \langle \partial^0 f_n(\tau), \psi(\tau)\rangle$ (as $n \to \infty$). Thus, the sequence $\langle \partial^0 f_n(\tau), \psi(\tau)\rangle$ converges for all $\psi \in \mathscr{D}((a, b))$. By [2], 315 Prop. 2 there exists $F \in \mathscr{D}'((a, b))$ such that $F = \lim \partial^0 f_n$. We show that $\{F\} = T$. For any $\varphi \in \mathscr{D}((-r, r))$ and any $t \in U_{|\varphi|}$, the equation

$$f_n(u) = T\delta_n(u) \qquad (t-|\varphi| \leqq u \leqq t+|\varphi|)$$

holds for n sufficiently large. Therefore,

$$\{F\}\varphi(t) = \langle F(\tau), \varphi(t-\tau)\rangle = \lim_{n\to\infty} \int_{t-|\varphi|}^{t+|\varphi|} f_n(u)\varphi(t-u)\,du = T\varphi(t).$$

Consequently $\{F\}=T$. If $T=0$ then, by (2), each $f_n=0$, from which it follows that $F=\lim \partial^0 f_n=0$. The mapping $F\to\{F\}$ is therefore one-to-one.

Characterization of distributions with left-bounded support

For the remainder of this paper we take $a=-\infty$. Accordingly, $\mathscr{D}((-r,r))=\mathscr{D}$ and $U_{|\varphi|}=(-\infty, b-|\varphi|)$ for all $\varphi\in\mathscr{D}$.

Definition. We denote by $P_+(-\infty,b)$ the subspace of all the elements T of $P(-\infty,b)$ such that for each φ in $\mathscr{D}$ the function $T\varphi$ vanishes to the left of some point (which depends on φ).

Theorem 3. *The mapping $F\to\{F\}$ is a linear bijection onto $P_+(-\infty,b)$ of the space of distributions in $\mathscr{D}'((-\infty,b))$ with left-bounded support.*

Proof. As a consequence of Theorem 2 we need only show that F has left-bounded support if and only if $\{F\}\in P_+(-\infty,b)$. Assume first that F has left-bounded support. Then there exists a number $\beta<b$ such that F vanishes on $(-\infty,\beta)$. Now, for any $\varphi\in\mathscr{D}$ and any $t\in U_{|\varphi|}$ it follows from $\operatorname{supp}\varphi\subset[-|\varphi|,|\varphi|]$ that

$$\{F\}\varphi(t) = \langle F(\tau), \varphi(t-\tau)\rangle = 0 \qquad (\text{all } t<\beta-|\varphi|),$$

from which we may conclude that $\{F\}\in P_+(-\infty,b)$. Assume now that $\{F\}\in\in P_+(-\infty,b)$. For $k=0,1,2,\dots$, we define X_k to be the set of all $\varphi\in\mathscr{D}([-1,1])$ such that $\{F\}_\varphi$ vanishes on the set $(-\infty,-k)\cap(-\infty,b-1)$. Then each X_k is a linear subspace of $\mathscr{D}([-1,1])$ and

$$\mathscr{D}([-1,1]) = \bigcup_{k=0}^{\infty} X_k. \tag{1}$$

Equation (1) is a consequence of the assumption $\{F\}\in P_+(-\infty,b)$. In addition, each X_k is closed; this follows from [2], 313, Prop. 1. The Fréchet space $\mathscr{D}([-1,1])$ is therefore the countable union of closed sets. By Baire's theorem [2], 213 some X_k must have a nonempty interior. But $X_k$is a linear subspace of $\mathscr{D}([-1,1])$ and therefore $\mathscr{D}([-1,1)=X_k$. Thus, if $\beta=\min(-k,b-1)$ we have $\{F\}\varphi(u)=0$ for all $u<\beta$ and all $\varphi\in\mathscr{D}([-1,1])$. In particular, if δ_n $(n=1,2,\dots)$ is a "δ-sequence" in $\mathscr{D}([-1,1])$, then

$$\{F\}\delta_n(u) = 0 \qquad (u<\beta,\ n=1,2,\dots). \tag{2}$$

Now, for any $\psi\in\mathscr{D}((-\infty,b))$ with $\operatorname{supp}\psi\subset(-\infty,\beta)$ there exist numbers c and d less than β such that $\operatorname{supp}\psi\subset[c,d]$. If we let $t=(1/2)(c+d)$ then $|\psi_t|<\beta-t$

and therefore, by (2), $\{F\}\delta_n(u)=0$ for all $u \leqq t+|\psi_t|$. Thus, from $\langle F(\tau), \psi(\tau)\rangle = \{F\}\psi_t(t)$ follows

$$\langle F(\tau), \psi(\tau)\rangle = \lim_{n\to\infty} \int_{t-|\psi_t|}^{t+|\psi_t|} \{F\}\delta_n(u)\psi_t(t-u)\,du = 0$$

recall that $(\{F\}\delta_n)^*\psi_t = (\{F\}\psi_t)^*\delta_n)$. Consequently, F has support bounded on the left by β.

Bibliography

[1] T. K. Boehme, Continuity and perfect operators, *J. London Math. Soc.* **39** (1964), 355—358.
[2] J. Horváth, Topological vector spaces and distributions I, *Reading*, 1966.
[3] H. Kestelman, Modern theories of integration, *New York*, 1960.
[4] G. Krabbe, Operational calculus, *New York*, 1970.
[5] L. Schwartz, Théorie des distributions, *Paris*, 1966.
[6] R. Struble, On operators and distributions, *Canad. Math. Bull.* **11** (1968), 61—64.
[7] R. Struble, An algebraic view of distributions and operators, *Studia Math.* **37** (1971), 103—109.
[8] F. Treves, Topological vector spaces, distributions and kernels. *New York*, 1967.
[9] A. Zemanian, Distribution theory and transform analysis, *New York*, 1965.
[10] H. Shultz, Algebraic approaches to distributions and operational calculus, Ph. D. Thesis, *Purdue University, Lafayette, Indiana*, 1970.

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