On the additive group of a torsion-free ring of rank two

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Throughout this note a group means an abelian group and a ring — an associative ring with an identity element. A ring is said to be torsion-free of rank two if its additive group is itself torsion-free of rank two.

A group G is said to admit a ring if there exists a ring (with identity) whose additive group is isomorphic to G.

If G is a torsion-free group $T(g)$ denotes the type[1]) of an element g in G and $\mathcal{T}(G)$ denotes the set of all types τ such that elements of type τ exist in G. $\mathcal{T}(G)$ is a partially ordered set. The type $\tau=(k_1, k_2, \dots, k_n, \dots)$ is said to be reduced if $k=0$ or ∞ for each i. A type τ^* is said to be maximal if there is no τ in $\mathcal{T}(G)$ such that $\tau>\tau^*$. A group G is homogeneous if all its non-zero elements are of the same type, in particular, all groups of rank one are homogeneous.

It is well known (see [4] or [2], 270) that a group of rank one admits a ring if and only if is type is reduced.

The main aim of this note is to prove some necessary conditions for a torsion-free group of rank two to admit a ring.

Theorem. *Let G be a torsion-free group of rank two. If G admits a ring R then $\mathcal{T}(G)$ contains at most three elements.*

PROOF. Since G admits a ring there exists a bilinear mapping $\varphi: G\times G \to G$ which can be factored through a mapping $\alpha: G\otimes G \to G$, where $G\otimes G$ denotes the which can be factored through a mapping $\alpha: G\otimes G \to G$, where $G\otimes G$ denotes the tensor product. It follows that Hom $(G\otimes G, G)$ and therefore also Hom $\big(G, \text{Hom}\,(G, G)\big)$ is non-trivial (see [3], 256).

We identify the elements of G with those of R. If g_0 is the identity of R, then for every g in G $T(g) = T(gg_0) = T\big(\alpha(g\otimes g_0)\big) \geqq T(g\otimes g_0) \geqq T(g_0)$. Also, since G is of rank two the elements of $\mathcal{T}(G)$ satisfy the maximum condition (see [1], 109). Indeed it follows from the argument there that in our case each chain is of length at most two. Therefore to prove our theorem it is enough to show that there are at most two maximal types.

Assume that g_1 and g_2 are elements in G with $T(g_1)=\tau_1$, $T(g_2)=\tau_2$, where τ_1 and τ_2 are maximal and $\tau_1\neq\tau_2$. Let G_i be a pure subgroup of G rank one contain-

[1]) For the definition of types see [2], 145—147.

ing g_i $(i=1, 2)$. (Such groups exist, see, for example, [3], 116). Consider the following exact sequence

$$0 \to G_1 \to G \to G/G_1 \to 0$$

Since G_2 is torsion-free it follows that

$$0 \to G_1 \otimes G_2 \to G \otimes G_2 \to G/G_1 \otimes G_2 \to 0$$

is also exact (see [3] p. 260).

Hence,

$$0 \to \mathrm{Hom}(G/G_1 \otimes G_2, G) \to \mathrm{Hom}(G \otimes G_2, G) \to \mathrm{Hom}(G_1 \otimes G_2, G)$$

is exact (see [3], 186).

Now, $\mathrm{Hom}(G_1 \otimes G_2, G) = 0$ since the type of every element of $G_1 \otimes G_2$ is greater than the type of any element in G. It follows that

$$\mathrm{Hom}(G/G_1 \otimes G_2, G) \cong \mathrm{Hom}(G \otimes G_2, G) \cong \mathrm{Hom}(G_2 \otimes G, G) \cong$$

$$\cong \mathrm{Hom}(G_2, \mathrm{Hom}(G, G)).$$

Suppose there exists an element g_3 in G of maximal type such that $T(g_3) \neq \tau_1, \tau_2$. Let $\bar{g}_3$ and $\bar{g}_2$ be the cosets of G_1 in G containing g_3 and g_2 respectively. Since G_1 is pure in G, G/G_1 is a torsion-free group of rank one and therefore homogeneous. So $T(\bar{g}_3) = T(\bar{g}_2)$. Also, $T(\bar{g}_3) \geqq T(g_3)$. Suppose $T(\bar{g}_3) = T(g_3)$. Then $T(g_3) = T(\bar{g}_2) \geqq T(g_2)$ which contradicts the maximality of $T(g_2)$. Therefore $T(\bar{g}_3) > T(g_3)$. By the maximality of $T(g_3)$, $\mathrm{Hom}(G/G_1 \otimes G_2, G) = 0$. Similarly

$$\mathrm{Hom}(G/G_2 \otimes G_1, G) = 0.$$

It follows that both $\mathrm{Hom}(G_1, \mathrm{Hom}(G, G))$ and $\mathrm{Hom}(G_2, \mathrm{Hom}(G, G))$ are trivial. Hence, $\mathrm{Hom}(G \otimes G, G) = 0$, but this contradicts the assumption that G admits a ring and completes the proof of the theorem.

Remarks. It is easy to see that

(i) R is commutative.

(ii) $T(g_0)$ is reduced, in particular if G is homogeneous then its type is reduced.

Further, if $T(G)$ contains three elements then they are all reduced. For, if g_1 and g_2 are elements of maximal type such that $T(g_1) \neq T(g_2)$, and $T(g_1)$ not reduced, then $T(g_1 \otimes g_2) > T(g_1)$ and $T(g_1 \otimes g_1) > T(g_1)$. It follows that $g_1 \cdot g_2 = g_1 \cdot g_1 = 0$. Hence $ng_1 = ng_1 \cdot g_0 = n_1 g_1 \cdot g_1 + n_2 g_1 \cdot g_2 = 0$ where n, n_1, n_2 are integers and $n \neq 0$, but this is a contradiction.

In the case of torsion-free groups of rank two the existence of a homomorphism $\alpha: G \otimes G \to G$ satisfying a certain condition ensures the existence of a ring structure on G.

Lemma. Let G be a torsion-free group of rank two. Then the following conditions are equivalent:

(i) G admits a ring.

(ii) There exists a homomorphism $\alpha: G \otimes G \to G$ and an element g_0 in G such that $\alpha(g_0 \otimes g) = \alpha(g \otimes g_0) = g$ for all g in G.

Proof. It is clear that (i) implies (ii). To show that (ii) implies (i), define $a\cdot b = \alpha(a\otimes b)$ for all a, b in G. Clearly this defines a multiplication on G which is distributive over addition, to show that it is a ring we need only prove the associative law.

Let g_1 be an element of G such that g_0 and g_1 are independent. Then

$$mg_1\cdot g_1 = m_0g_0+m_1g_1,$$

where m, m_0, m_1 are integers.

Therefore

$$m(g_1\cdot g_1)\cdot g_1 = m_0g_0\cdot g_1 + m_1g_1\cdot g_1 = m_0g_1\cdot g_0+m_1g_1\cdot g_1 = mg_1\cdot(g_1\cdot g_1).$$

Hence,

$$(g_1\cdot g_1)\cdot g_1 = g_1\cdot(g_1\cdot g_1).$$

It follows that $(a\cdot b)\cdot c = a\cdot(b\cdot c)$ if each of the elements a, b and c is equal either to g_0 or g_1. Since every g in G satisfies an equation of the form $ng = n_0g_0+n_1g_1$ for some integers n, n_0, n_1, and since the distributive law holds it follows that the multiplication defined above is associative. This completes the proof of the lemma.

References

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