Discret distribution and permanents

Dedicated to Professor A. Rapcsák on the occasion of his 60 th Birthday

By BÉLA GYIRES (Debrecen)

§ 1. Introduction

Although the notion of permanent goes back to Cauchy ([3], 577), permanents have never attracted as much attention as e.g. determinants. Several investigations have dealt of late with the permanents of doubly stochastic matrices. The reason for this is probably to be found in van der Waerden's conjecture formulated back in 1926 ([6]), and not yet solved. If permanents never occupied the centre of the stage, this can be attributed mainly to the fact that hardly any applications could be found for them.

One of the aims of the present paper is to point out applications of permanents in probability theory.

In § 3. we employ the Cauchy—Binet expansion for permanents of product matrices in order to define discretely distributed random vector-variables generalizing the polynomial distribution.

In § 4. we obtain equalities and inequalities for permanents, starting from probabilistic models. Among the results obtained, a new procedure for the approximative determination of the eigenvalues of positive definite or semidefinite Hermitian matrices, based on the computation of permanents merits to be mentioned. In this same section it is also shown that the application of the method of Graeffe—Bernoulli for determining the greatest eigenvalue of positive semidefinite Hermite-symmetrical matrices, and the new procedure above mentioned stem from a common root.

In § 2. we consider results needed in the sequel, in particular the Cauchy—Binet expansion for the permanent of a product matrix, as well as some theorems concerning the moments of discrete random variables.

§ 2. Preliminaries

a) Let the elements of the matrix

$$A = (a_{jk}) \qquad (j, k = 1, \dots, m)$$

be complex numbers. By the trace of this matrix we mean the expression

$$\operatorname{tr} A = \sum_{j=1}^{m} a_{jj},$$

and by its permanent the expression

$$\operatorname{Per} A = \sum a_{1k1} \dots a_{mkm}, \qquad (k_1, \dots, k_m) \in P_m$$

where P_m is the set of permutations without repetition of the elements $1, \dots, m$.

Let A and B be matrices of m rows and r columns. We denote by $A^{(k)}_{\beta_k}$ the matrix with β_k columns, each of them equal to the k-th column of the matrix A. Again $(A^{(1)}_{\beta_1} \dots A^{(r)}_{\beta_r})$ denotes the matrix obtained by writing the matrices $A^{(1)}_{\beta_1}, \dots, A^{(r)}_{\beta_r}$ consecutively, and having thus m rows and $\beta_1 + \dots + \beta_r$ columns. If $\beta_k = 0$, this means that from the matrix $(A^{(1)}_{\beta_1} \dots A^{(r)}_{\beta_r})$ the k-th column of the matrix A is missing. Let B^T denote the transpose of the matrix B.

In the sequel we shall often use the Cauchy—Binet expansion theorem for the permanent of matrix products ([3], 579). According to this theorem

$$(1) \qquad \operatorname{Per}(AB^T) = \sum \frac{1}{\beta_1! \dots \beta_r!} \operatorname{Per}(A^{(1)}_{\beta_1} \dots A^{(r)}_{\beta_r}) \operatorname{Per}(B^{(1)}_{\beta_1} \dots B^{(r)}_{\beta_r}),$$

where the summation extends to the integers satisfying the conditionsons

$$(2) \qquad 0 \leqq \beta_k \leqq m \qquad (k = 1, \dots, r), \; \beta_1 + \dots + \beta_r = m.$$

Let

$$\Lambda = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_r \end{pmatrix},$$

where $\lambda_1, \dots, \lambda_r$ are complex numbers. From the identity (1) we get

$$(3) \qquad \operatorname{Per}(A\Lambda B^T) = \sum \frac{\lambda_1^{\beta_1} \dots \lambda_r^{\beta_r}}{\beta_1! \dots \beta_r!} \operatorname{Per}(A^{(1)}_{\beta_1} \dots A^{(r)}_{\beta_r}) \operatorname{Per}(B^{(1)}_{\beta_1} \dots B^{(r)}_{\beta_r}).$$

b) In proving our statements formulated in § 2., we shall need the following results:

Lemma 1. *If the random variable ξ is definied by*

$$P(\xi = a_k) = p_k \quad (k = 1, \dots, n), \qquad \sum_{k=1}^{n} p_k = 1,$$

where a_k $(k = 1, \dots, n)$ are complex numbers and if

$$(4) \qquad M_{\alpha\beta}(\xi) = \sum_{k=1}^{n} a_k^{\alpha} \bar{a}_k^{\beta} p_k,$$

then the matrices

$$(5) \qquad \mathcal{M}_\nu(\xi) = \mathcal{M}_\nu = \begin{pmatrix} M_{00} & M_{01} & \dots & M_{0\nu} \\ M_{10} & M_{11} & \dots & M_{1\nu} \\ \cdot & \cdot & \dots & \cdot \\ M_{\nu 0} & M_{\nu 1} & \dots & M_{\nu\nu} \end{pmatrix} \qquad (\nu = 0, 1, 2, \dots)$$

Hermite-symmetric in view of $M_{\alpha\beta} = \overline{M}_{\beta\alpha}$, are positive semidefinite matrices.

PROOF. For any nonnegative integer v and for arbitrary complex numbers $x_0, x_1, \ldots, x_v$ the relation

$$\sum_{k=1}^{n} p_k |x_0 + a_k x_1 + \cdots + a_k^v x_v|^2 = \sum_{\alpha=0}^{v} \sum_{\beta=0}^{v} \left(\sum_{k=1}^{n} a_k^\alpha \bar{a}_k^\beta p_k \right) x_\alpha \bar{x}_\beta \geqq 0$$

holds, and in view of (4) this already proves our statement.

Lemma 2. *If besides the conditions of Lemma 1. one also has a_k $(k=1, \ldots, n)$ real and*

$$M_\alpha = E(\xi^\alpha) \qquad (\alpha = 0, 1, 2, \ldots),$$

then the Hankel-symmetrical matrices

$$\mathcal{M}_v(\xi) = \mathcal{M}_v = \begin{pmatrix} M_0 & M_1 & \ldots & M_v \\ M_1 & M_2 & \ldots & M_{v+1} \\ \cdot & \cdot & \ldots & \cdot \\ M_v & M_{v+1} & \ldots & M_{2v} \end{pmatrix} \qquad (v = 0, 1, 2, \ldots) \tag{6}$$

are positive semidefinite and

$$\operatorname{Det} \mathcal{M}_v(\xi) = 0 \qquad (v = n+1, n+2, \ldots). \tag{7}$$

PROOF. From the fact that (4) implies $M_{\alpha\beta} = M_{\alpha+\beta}$, there follows the positive semidefiniteness of $\mathcal{M}_v$. Since the distribution function of the discrete random variable ξ can have at most n jumps, (7) is a consequence of a theorem due to Hamburger ([4], 248).

Lemma 3. *If besides the conditions of Lemma 1. the numbers a_k $(k=1, \ldots, n)$ satisfy also the conditions*

$$a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0, \qquad a_1 > 0, \; p_1 > 0, \tag{8}$$

then

$$\frac{M_{k+1}}{M_k} \uparrow a_1, \qquad k \to \infty, \tag{9}$$

and

$$\sqrt[k]{M_k} \uparrow a_1, \qquad k \to \infty.$$

PROOF. Condition (8) implies $M_k > 0$ $(k = 0, 1, \ldots)$ and so the sequence

$$\frac{M_{k+1}}{M_k} \qquad (k = 0, 1, 2, \ldots) \tag{10}$$

exists. Since the matrices (6) are positive semidefinite,

$$\operatorname{Det} \begin{pmatrix} M_{2k(-1)} & M_{2k-1} \\ M_{2k-1} & M_{2k} \end{pmatrix} \geqq 0,$$

i. e.

$$M_{2k-1}^2 \leqq M_{2(k-1)} M_{2k} \qquad (k = 1, 2, \ldots). \tag{11}$$

Let now be $\eta = +\sqrt{\xi}$. Since the Hankel-matrices

$$\mathcal{M}_\nu(\eta) \qquad (\nu = 0, 1, 2, \ldots)$$

formed with the help of the moments

$$E(\eta^k) = M'_k \qquad (k = 0, 1, 2, \ldots)$$

are also positive semidefinite, and since

$$M'_{2k} = M_k \qquad (k = 0, 1, 2, \ldots),$$

we have

$$\operatorname{Det}\begin{pmatrix} M'_{4k-2} & M'_{4k} \\ M'_{4k} & M_{4k+2} \end{pmatrix} = \operatorname{Det}\begin{pmatrix} M_{2k-1} & M_{2k} \\ M_{2k} & M_{2k+1} \end{pmatrix} \geqq 0,$$

i.e.

$$M_{2k}^2 \leqq M_{2k-1} M_{2k+1}. \tag{12}$$

It follows from the inequalities (11) and (12) that the sequence (10) is monotone nondecreasing.

In view of

$$M_k = \sum_{j=1}^n a_j^k p_j = a_1^k \sum_{j=1}^n \left(\frac{a_j}{a_1}\right)^k p_j,$$

in the factor of a_1^k on the right hand side all the members containing factors $0 \leqq \frac{a_j}{a_1} < 1$ tend to zero for $k \to \infty$. The remaining positive members (by (8) there exists at least one such member) are independent of k. This proves the validity of our statement (9).

That the sequence

$$\sqrt[k]{M_k} \qquad (k = 1, 2, 3, \ldots) \tag{13}$$

is also monotone nondecreasing follows from the Hölder's inequality and is a well-known fact of probability theory ([1], 68, lemma 1.). On the other hand, $\sqrt[k]{M_k}$ being the geometric mean of the first k terms of the sequence (10), the sequence (13) tends also to the limit a_1.

§ 3. Definition of random-vector-variables with the help of permanents

Starting with the Cauchy—Binet identity exposed in § 2. a.), we define in this section random vector-variables on nonnegative integers, we determine their characteristic function, expectation and matrix of covariance.

a) Let the matrices with nonnegative elements

$$A = (a_{jk}), \quad B = (b_{jk}) \qquad (j = 1, \ldots, m;\ k = 1, \ldots, r)$$

satisfy condition

$$AB^T = M, \tag{14}$$

where M is the matrix having all its elements equal to 1. By the representation (1) one has

$$\sum \frac{1}{\beta_1! \dots \beta_r!} \operatorname{Per}(A^{(1)}_{\beta_1} \dots A^{(r)}_{\beta_r}) \operatorname{Per}(B^{(1)}_{\beta_1} \dots B^{(r)}_{\beta_r}) = m!.$$

Since by assumption

$$\operatorname{Per}(A^{(1)}_{\beta_1} \dots A^{(r)}_{\beta_r}) \geqq 0, \quad \operatorname{Per}(B^{(1)}_{\beta_1} \dots B^{(r)}_{\beta_r}) \geqq 0,$$

the following definition makes sense:

Definition 1. *We say that the random vector-variable* $\xi^T=(\xi_1, \dots, \xi_r)$ *has polynomial distribution generated by the matrices A and B, if*

$$(15) \quad P(\xi_1=\beta_1, \dots, \xi_r=\beta_r) = \frac{1}{m!\,\beta_1! \dots \beta_r!} \operatorname{Per}(A^{(1)}_{\beta_1} \dots A^{(r)}_{\beta_r}) \operatorname{Per}(B^{(1)}_{\beta_1} \dots B^{(r)}_{\beta_r}),$$

where the integers $\beta_1, \dots, \beta_r$ *satisfy the conditions* (2).

Theorem 1. *If* $\varphi(t_1, \dots, t_r)$ *denotes the characteristic function of the random vector-variable* ξ *having polynomial distribution generated by the matrices A and B, then*

$$(16) \qquad \varphi(t_1, \dots, t_r) = \frac{1}{m!} \operatorname{Per} A \begin{pmatrix} e^{it_1} & & (0) \\ & \ddots & \\ (0) & & e^{it_r} \end{pmatrix} B^T.$$

PROOF. By definition we have

$$\varphi(t_1, \dots, t_r) = E(e^{i(t_1\xi_1 + \dots + t_r\xi_r)}) =$$
$$= \frac{1}{m!} \sum \frac{e^{i(t_1\beta_1 + \dots + t_r\beta_r)}}{\beta_1! \dots \beta_r!} \operatorname{Per}(A^{(1)}_{\beta_1} \dots A^{(r)}_{\beta_r}) \operatorname{Per}(B^{(1)}_{\beta_1} \dots B^{(r)}_{\beta_r})$$

and on the basis of the identity (3) this yields the expression (16).

On the basis of (16) we are able to compute without much difficulty the expectation-vector and the moments of second order of a random vector-variable. Indeed, by (16)

$$E(\xi_k) = \frac{1}{i}\left[\frac{\partial}{\partial t_k} \varphi(t_1, \dots, t_r)\right]_{t_1 = \dots = t_r = 0} = \frac{1}{m!} \sum_{j=1}^{m} \operatorname{Per} \begin{pmatrix} 1 \dots 1 & a_{1k} b_{jk} & 1 \dots 1 \\ \cdots & \cdots & \cdots \\ 1 \dots 1 & a_{mk} b_{jk} & 1 \dots 1 \end{pmatrix},$$

and so

$$E(\xi_k) = \frac{1}{m} \sum_{j=1}^{m} a_{jk} \sum_{j=1}^{m} b_{jk} \qquad (k=1, \dots, r).$$

Again, a bit of computation will yield

$$E(\xi_k \xi_l) = \delta_{kl} E(\xi_k) +$$
$$+ \frac{1}{m(m-1)} \left(\sum_{j=1}^{m} a_{jk} \sum_{j=1}^{m} a_{jl} - \sum_{j=1}^{m} a_{jk} a_{jl} \right) \left(\sum_{j=1}^{m} b_{jk} \sum_{j=1}^{m} b_{jl} - \sum_{j=1}^{m} b_{jk} b_{jl} \right),$$

where δ_{kl} is the Kronecker symbol.

Since $\xi_1+\dots+\xi_r = m$, the matrix cov ξ has the sum of each of its rows equal to zero, and the distribution of ξ is degenerate.

Corollary 1. If each element of the matrix B is equal to 1, *and if the matrix A with nonnegative elements satisfies also the conditions*

$$\sum_{k=1}^{r} a_{jk} = 1 \qquad (j=1, \dots, m),$$

then

$$P(\xi_1=\beta_1, \dots, \xi_r=\beta_r) = \frac{1}{\beta_1 ! \dots \beta_r !} \operatorname{Per}(A_{\beta_1}^{(1)} \dots A_{\beta_r}^{(r)}), \tag{17}$$

where $\beta_1, \dots, \beta_r$ satisfy conditions (2) *and*

$$\varphi(t_1, \dots, t_r) = \prod_{j=1}^{m} (a_{j1} e^{it_1} + \dots + a_{jr} e^{it_r}).$$

Moreover

$$E(\xi_k) = \sum_{j=1}^{m} a_{jk} \qquad (k=1, \dots, r)$$

and

$$\operatorname{cov} \xi = \begin{pmatrix} E(\xi_1) & & (0) \\ & \ddots & \\ (0) & & E(\xi_r) \end{pmatrix} - A^T A.$$

The statements of Corollary 1. now follow from the fact that under the hypotheses made (14) is automatically fulfilled and

$$\operatorname{Per}(B_{\beta_1}^{(1)} \dots B_{\beta_r}^{(r)}) = m!,$$

i.e. (17) is in fact a distribution. The remaining formulae can be found by substituting into the formulae already obtained.

From Corollary 1. we obtain the following

Corollary 2. If besides the requirements of Corollary 1. $r=m$, and $A=S$ is a doubly stochastic matrix, then

$$E(\xi_k) = 1 \qquad (k=1, \dots, m), \ \operatorname{cov} \xi = I - S^T S$$

(where I is the unit matrix).

In this connection it is perhaps worth while to remark that on the basis of (17) we have for any doubly stochastic matrix S

$$\sum \frac{1}{\beta_1 ! \dots \beta_m !} \operatorname{Per}(S_{\beta_1}^{(1)} \dots S_{\beta_m}^{(m)}) = 1,$$

where the summation extends to the integers $\beta_1, \dots, \beta_m$ satisfying (19). In this formula all $(S_{\beta_1}^{(1)} \dots S_{\beta_m}^{(m)})$ are stochastic matrices.

Corollary 3. If besides the conditions of Corollary 1. the equalities

$$a_{jk} = a_k \qquad (j=1, \dots, m;\ k=1, \dots, r)$$

also hold, then

$$P(\xi_1=\beta_1, \dots, \xi_r=\beta_r) = \frac{m!}{\beta_1! \dots \beta_r!} a_1^{\beta_1} \dots a_r^{\beta_r},$$

i.e. one has a polynomial distribution. Moreover

$$\varphi(t_1, \dots, t_r) = (a_1 e^{it_1} + \dots + a_r e^{it_r})^m,$$

$$E(\xi_k) = ma_k \qquad (k=1, \dots, r),$$

$$\operatorname{cov} \xi = m \begin{pmatrix} a_1 & & (0) \\ & \ddots & \\ (0) & & a_r \end{pmatrix} - A^T A.$$

These well-known formulae are obtained from those of Corollary 1. partly by straightforward substitution, partly with the help of the equality

$$\operatorname{Per}(A_{\beta_1}^{(1)} \dots A_{\beta_r}^{(r)}) = m! a_1^{\beta_1} \dots a_r^{\beta_r}$$

valid by virtue of the surplus conditions.

b) Let the unitary matrix

$$\mathcal{U} = (a_{jk}) \qquad (j, k=1, \dots, m),\ \mathcal{U}^* \mathcal{U} = I$$

be given. $\mathcal{U}^*$ denotes the conjugate of the transpose of the matrix $\mathcal{U}$. On the basis of (1)

$$\operatorname{Per} \mathcal{U}^* \mathcal{U} = \sum \frac{1}{\beta_1! \dots \beta_m!} |\operatorname{Per}(\mathcal{U}_{\beta_1}^{(1)} \dots \mathcal{U}_{\beta_m}^{(m)})|^2 = 1, \tag{18}$$

where the summation extends to the nonnegative integers satisfying

$$0 \leqq \beta_k \leqq m \qquad (k=1, \dots, m),\ \beta_1 + \dots + \beta_m = m. \tag{19}$$

On the right hand side of (18) there are clearly m^m summands.

Definition 2. *The random vector-variable $\xi^T = (\xi_1, \dots, \xi_m)$ is said to have polynomial distribution generated by the unitary matrix $\mathcal{U}$, if*

$$P(\xi_1=\beta_1, \dots, \xi_m=\beta_m) = \frac{1}{\beta_1! \dots \beta_m!} |\operatorname{Per}(\mathcal{U}_{\beta_1}^{(1)} \dots \mathcal{U}_{\beta_m}^{(m)})|^2,$$

where the integers $\beta_1, \dots, \beta_m$ satisfy the conditions (9)1.

Theorem 2. *The characteristic function of the polynomial distribution generated by the unitary matrix $\mathcal{U}$ is equal to the permanent of the unitary matrix*

$$\mathcal{U}^* \begin{pmatrix} e^{it_1} & & (0) \\ & \ddots & \\ (0) & & e^{it_m} \end{pmatrix} \mathcal{U} \tag{20}$$

Moreover

$$E(\xi_k) = 1 \qquad (k=1, \dots, m) \tag{21}$$

and

$$\operatorname{cov} \xi = I - SS^T, \tag{22}$$

where

$$S = \begin{pmatrix} |u_{11}|^2 & \dots & |u_{1m}|^2 \\ \cdots & \cdots & \cdots \\ |u_{m1}|^2 & \dots & |u_{mm}|^2 \end{pmatrix}$$

is a doubly stochastic matrix.

PROOF. Let us take into account the definition of the characteristic function $d(t_1, \dots, t_m)$ of the random vector-variable ξ, as well as the manner in which its φistribution was introduced by (18). If we still consider the representation (3), we see that this characteristic function is in fact the permanent of the matrix (20). The formula for the matrix (20) trivially shows that this matrix is unitary.

If we perform the multiplications in the matrix (20), we obtain

$$\varphi(t_1, \dots, t_m) = \operatorname{Per} \begin{pmatrix} s_{11} & s_{12} & \dots & s_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ s_{m1} & s_{m2} & \dots & s_{mm} \end{pmatrix},$$

where

$$s_{kl} = \sum_{j=1}^{m} a_{kj} \bar{a}_{lj} e^{it_j} = s_{kl}(t_1, \dots, t_m).$$

Thus

$$\frac{\partial}{\partial t_j} \varphi(t_1, \dots, t_m) = i \sum_{k=1}^{m} \bar{a}_{kj} e^{it_j} \operatorname{Per} \begin{pmatrix} s_{11} & \dots & s_{1\,k-1} & a_{1j} & s_{1\,k+1} & \dots & s_{1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ s_{m1} & \dots & s_{m\,k-1} & a_{mj} & s_{m\,k+1} & \dots & s_{mm} \end{pmatrix},$$

hence by $s_{kl}(0, \dots, 0) = \delta_{kl}$ we get

$$E(\xi_k) = \frac{1}{i} \left[\frac{\partial}{\partial t_k} \varphi(t_1, \dots, t_m) \right]_{t_1 = \dots = t_m = 0} = \sum_{j=1}^{m} |a_{kj}|^2 = 1 \qquad (k=1, \dots, m)$$

in comformity with (21).

Similarly

$$E(\xi_k \xi_l) = - \left[\frac{\partial^2}{\partial t_k \, \partial t_l} \varphi(t_1, \dots, t_m) \right]_{t_1 = \dots = t_m = 0} = \delta_{kl} \sum_{\alpha=1}^{m} |a_{\alpha k}|^2 + \sum_{\substack{\alpha, \beta = 1 \\ \alpha \neq \beta}}^{m} |a_{\alpha k}|^2 |a_{\beta l}|^2$$

and now a simple computation yields

$$E(\xi_k \xi_l) = 1 + \delta_{kl} - \sum_{\alpha=1}^{m} |a_{\alpha k}|^2 |a_{\alpha l}|^2.$$

From this we get (22) without difficulty.

If we compare our result just obtained with the results in a), Corollary 2. concerning the polynomial distribution generated by doubly stochastic matrices, we see that the expectation-vector and the covariance matrix coincide for the two distributions, if the generating doubly stochastic matrix is S. Nevertheless, the two distributions fail to be identical, as is shown by the fact that their characteristic functions are different.

Here too it is easy to verify that Det cov $\xi=0$, i.e. the polynomial distributions generated by unitary matrices are degenerate.

§ 4. On the matrices of permanents

In this section we are going to deal with the Hankelian matrices of permanents. The method to be followed will be to start with some probability distribution as a model, and to apply to the moments of this distribution the lemmas formulated in § 2. b.).

a) We have the following

Theorem 3. *Let A and B be matrices of m rows and r columns with nonnegative elements and satisfying $AB^T=M$. Let moreover Λ be the diagonal matrix with real elements $\lambda_1, \ldots, \lambda_r$. Then the Hankelain matrices* (6) *formed with the help of the quantities*

$$(23)\qquad M_k = \frac{1}{m!}\operatorname{Per}(A\Lambda^k B^T) \qquad (k=0, 1, 2, \ldots)$$

are positive semidefinite symmetrical matrices, and

$$(24)\qquad \operatorname{Det} \mathcal{M}_\nu = 0, \quad \nu \geqq r^m+1.$$

Proof. On the basis of the probabilities (15), let the random variable ξ be defined by

$$(25)\qquad P(\xi=\lambda_1^{\beta_1}\ldots\lambda_r^{\beta_r}) = \frac{1}{m!\,\beta_1!\ldots\beta_r!}\operatorname{Per}(A_{\beta_1}^{(1)}\ldots A_{\beta_r}^{(r)})\operatorname{Per}(B_{\beta_1}^{(1)}\ldots B_{\beta_r}^{(r)}),$$

where the integers $\beta_1, \ldots, \beta_r$ satisfy conditions (2). Since by (25)

$$E(\xi^k) = \sum\frac{(\lambda_1^{\beta_1}\ldots\lambda_r^{\beta_r})^k}{m!\,\beta_1!\ldots\beta_r!}\operatorname{Per}(A_{\beta_1}^{(1)}\ldots A_{\beta_r}^{(r)})\operatorname{Per}(B_{\beta_1}^{(1)}\ldots B_{\beta_r}^{(r)}),$$

we get on the basis of (7)

$$E(\xi^k) = \frac{1}{m!}\operatorname{Per}(A\Lambda^k B^T) = M_k$$

and so by virtue of Lemma 2. the Hankelian matrix (6) formed with the help of the quantities (23) is in fact a positive semidefinite symmetrical matrix. Since, moreover, there are not more than r^m numbers $\lambda_1^{\beta_1}\ldots\lambda_r^{\beta_r}$ parwise different and satisfying conditions (2), again on the basis of Lemma 2. (24) also holds.

Corollary 4. If $\lambda_1, \ldots, \lambda_r$ are real numbers and

$$a_{jk} \geqq 0 \qquad (j=1, \ldots, m;\ k=1, \ldots, r),\ \sum_{k=1}^{r} a_{jk} = 1,$$

then the Hankelian matrices (6) *formed with the help of the quantities*

$$M_k = \prod_{j=1}^{m} (a_{j1}\lambda_1^k + \cdots + a_{jr}\lambda_r^k) \qquad (k=0, 1, 2, \dots)$$

are positive semidefinite symmetrical matrices, and

$$\mathrm{Det}\, \mathcal{M}_\nu = 0, \qquad \nu \geqq r^m + 1.$$

Proof. If in conformity with what has been mentioned in Corollary 1. the random variable ξ is defined by

$$P(\xi = \lambda_1^{\beta_1} \dots \lambda_r^{\beta_r}) = \frac{1}{\beta_1! \dots \beta_r!} \mathrm{Per}\,(A_{\beta_1}^{(1)} \dots A_{\beta_r}^{(r)}),$$

where $\beta_1, \dots, \beta_r$ satisfy (2), then it is easy to see that

$$E(\xi^k) = M_k$$

and so on the basis of Lemma 2. our statement does in fact hold.

Corollary 5. If $\lambda_1, \dots, \lambda_r$ are real, m is a natural number and

$$a_k \geqq 0 \qquad (k=1, \dots, r), \; \sum_{k=1}^{r} a_k = 1,$$

then the Hankelian matrices (6) *formed with the help of the quantities*

$$M_k = (a_1\lambda_1^k + \cdots + a_r\lambda_r^k)^m \qquad (k=0, 1, 2, \dots) \tag{26}$$

are positive semidefinite symmetric matrices, and

$$\mathrm{Det}\, \mathcal{M}_\nu = 0, \qquad \nu \geqq r^m + 1.$$

Proof. Follows directly from Corollary 4.

It is perhaps worth while to note that the expressions (26) are the moments of the random variable defined by

$$P(\xi = \lambda_1^{\beta_1} \dots \lambda_r^{\beta_r}) = \frac{1}{\beta_1! \dots \beta_r!} a_1^{\beta_1} \dots a_r^{\beta_r}.$$

In particular

$$E(\xi) = \left(\sum_{k=1}^{r} a_k \lambda_k\right)^m,$$

$$D^2(\xi) = \left(\sum_{k=1}^{r} a_k \lambda_k^2\right)^m - \left(\sum_{k=1}^{r} a_k \lambda_k\right)^{2m}.$$

If $m=1$, the above distribution reduced to the discrete distribution $P(\xi=\lambda_j)=a_j$.

On the basis of Corollary 5. we have the following

Theorem 4. *If the eigenvalues of the quadratic matrix A of order m are real, then the Hankelian matrices* (6) *formed with the help of the quantities*

$$M_k = \mathrm{tr}\, A^k \qquad (k=0, 1, 2, \dots)$$

are positive semidefinite symmetrical matrices, and

$$\operatorname{Det} \mathcal{M}_\nu = 0, \qquad \nu \geqq m+1.$$

Theorem 5. *If A is a normal matrix of order m, then the Hermitian symmetric matrices* (5) *formed with the help*

$$M_{\alpha\beta} = \operatorname{Per}(A^\alpha A^{*\beta}) \qquad (\alpha, \beta = 0, 1, 2, \dots) \tag{27}$$

are positive semidefinite matrices.

PROOF. Since A is a normal matrix, it can be represented in the form

$$A = \mathcal{U}^* \Lambda \mathcal{U}, \qquad \mathcal{U}^* \mathcal{U} = I,$$

where Λ is the diagonal matrix formed from the eigenvalues $\lambda_1, \dots, \lambda_m$ of the matrix A. In view of (18), we can define the random variable ξ by

$$P(\xi = \lambda_1^{\beta_1} \dots \lambda_m^{\beta_m}) = \frac{1}{\beta_1! \dots \beta_m!} |\operatorname{Per}(\mathcal{U}_{\beta_1}^{(1)} \dots \mathcal{U}_{\beta_m}^{(m)})|^2, \tag{28}$$

where $\beta_1, \dots, \beta_m$ are integers satisfying the conditions (19). Since the quantities (4) belonging to the distribution (28) are given by

$$M_{\alpha\beta}(\xi) = \sum (\lambda_1 \dots \lambda_m)^\alpha (\bar{\lambda}_1 \dots \bar{\lambda}_m)^\beta \frac{1}{\beta_1! \dots \beta_m!} |\operatorname{Per}(\mathcal{U}_{\beta_1}^{(1)} \dots \mathcal{U}_{\beta_m}^{(m)})|^2,$$

the representation (3) implies

$$M_{\alpha\beta}(\xi) = M_{\alpha\beta}.$$

Thus by Lemma 1. the Hermitian symmetric matrices formed with the help of the quantities (27) are in fact positive semidefinite matrices.

Corollary 6. If S is a doubly stochastic normal matrix of order m, then the symmetric matrices (5) *formed with the help of the quantities*

$$M_{\alpha\beta} = \operatorname{Per}(S^\alpha S^{T\beta}) \qquad (\alpha, \beta = 0, 1, 2, \dots)$$

are positive semidefinite matrices.

PROOF. On the basis of Theorem 5. our statement is trivial.

It is interesting in connection with this corolary, that by our hypotheses $S^T S^{T\beta}$ is a stochastic matrix, and so $0 < M_{\alpha\beta} \leqq 1$.

Corollary 7. If A is a Hermitian symmetric matrix, then the symmetric matrices (6) *formed with the help of the quantities*

$$M_k = \operatorname{Per} A^k \qquad (k = 0, 1, 2, \dots)$$

are positive semidefinite matrices and

$$\operatorname{Det} \mathcal{M}_\nu = 0, \qquad \nu \geqq m^m + 1.$$

PROOF. Since by hypothesis $A^* = A$, our statement immediately follows from Theorem 5. and from the proof of Theorem 3.

b) The following theorem renders possible the numerical determination of the greatest eigenvalue of positive semidefinite Hermitian symmetric matrices.

Theorem 6. *If the Hermitian symmetric positive semidefinite matrix A has its greatest eigenvalue λ_1 positive, then*

$$\frac{\operatorname{Per} A^{k+1}}{\operatorname{Per} A^k} \uparrow \lambda_1^m, \qquad k \to \infty$$

and

$$\sqrt[k]{\operatorname{Per} A^k} \uparrow \lambda_1^m, \qquad k \to \infty. \tag{29}$$

Proof. The proof of the theorem follows immediately on the basis of Corollary 7. and with the help of Lemma 3.

The statement expressed by (29) can be found also in the paper [2] (Th. 11) under the condition that the greatest eigenvalue has multiplicity one.

As is known, by the ν-th ($\nu=1, \dots, m$) derivate of a matrix A of order m we understand the matrix A_ν of order $\binom{m}{n}$ the elements of which are the signed deteminants of oıder ν which can be formed from the matrix A, the rows and the columns being arranged lexicographically. It is known that the eigenvalues of A_ν are the combinational products without repetition of order ν of the eigenvalues of A.

On the basis of Theorem 6. one has the following

Theorem 7. *If the eigenvalues of the Hermitian symmetric positive definite matrix A of order m are*

$$\lambda_1 \geqq \dots \geqq \lambda_m > 0, \tag{30}$$

then

$$\sqrt[k]{\operatorname{Per} A_\nu^k} \uparrow (\lambda_1 \dots \lambda_\nu)^{\binom{m}{\nu}}, \qquad k \to \infty, \tag{31}$$

$$\frac{\operatorname{Per} A_\nu^{k+1}}{\operatorname{Per} A_\nu^k} \uparrow (\lambda_1 \dots \lambda_\nu)^{\binom{m}{\nu}}, \qquad k \to \infty, \ (\nu=1, \dots, m). \tag{32}$$

Proof. Follows immediately from Theorem 6.

Theorem 8. *If A is a Hermitian symmetric positive definite matrix of order m, then*

$$\lim_{k\to\infty} \sqrt[k]{\frac{\operatorname{Per} A_\nu^k}{\operatorname{Per} A_{m-\nu}^{-k}}} = (\operatorname{Det} A)^{\binom{m}{\nu}}, \tag{33}$$

$$\lim_{k\to\infty} \frac{\operatorname{Per} A_\nu^{k+1} \operatorname{Per} A_{m-\nu}^{-k}}{\operatorname{Per} A_\nu^k \operatorname{Per} A_{m-\nu}^{-(k+1)}} = (\operatorname{Det} A)^{\binom{m}{\nu}}. \tag{34}$$

Proof. Under the condition (30) the eigenvalues of A^{-1} are

$$\frac{1}{\lambda_m} \geqq \dots \geqq \frac{1}{\lambda_1} > 0$$

and so by Theorem 7.

$$(35)\qquad \sqrt[k]{\operatorname{Per} A_{m-\nu}^{-k}} \uparrow \left(\frac{1}{\lambda_m \dots \lambda_{\nu+1}}\right)^{\binom{m}{\nu}}, \qquad k \to \infty,$$

$$(36)\qquad \frac{\operatorname{Per} A_{m-\nu}^{-(k+1)}}{\operatorname{Per} A_{m-\nu}^{-k}} \uparrow \left(\frac{1}{\lambda_m \dots \lambda_{\nu+1}}\right)^{\binom{m}{\nu}}, \qquad k \to \infty.$$

If we divide the sequence (31) by the sequence (35), and the sequence (32) by (36), then passage to the limit yields — in comformity with our statement — formulas (33) and (34).

Formulae (31) and (32) remind us of the wellknown Graeffe—Bernoulli procedure for root-approximation. ([5], 36.)

As a matter of fact, on the basis of Theorem 4. and of Lemma 3. there holds the Graeffe—Bernoulli theorem for the determination of the greatest eigenvalue of matrices.

Theorem 9. *If the greatest eigenvalue λ_1 of the matrix A of order m is positive, while the other eigenvalues are nonnegative, then*

$$\frac{\operatorname{tr} A^{k+1}}{\operatorname{tr} A^k} \uparrow \lambda_1, \qquad k \to \infty$$

and

$$\sqrt[k]{\operatorname{tr} A^k} \uparrow \lambda_1, \qquad k \to \infty.$$

Of course, the theorem corresponding to the Theorem 7. and 8. is also valid.

Theorem 10. *If the eigenvalues of the matrix A of order m are positive and their increasing order is given by* (3), *then*

$$\sqrt[k]{\operatorname{tr} A_\nu^k} \uparrow \lambda_1 \dots \lambda_\nu, \qquad k \to \infty,$$

and

$$\frac{\operatorname{tr} A_\nu^{k+1}}{\operatorname{tr} A_\nu^k} \uparrow \lambda_1 \dots \lambda_\nu, \qquad k \to \infty.$$

Theorem 11. *If the eigenvalues of the matrix A of order m are positive numbers, then*

$$\lim_{k \to \infty} \sqrt[k]{\frac{\operatorname{tr} A_\nu^k}{\operatorname{tr} A_{m-\nu}^{-k}}} = \operatorname{Det} A$$

and

$$\lim_{k \to \infty} \frac{\operatorname{tr} A_\nu^{k+1} \operatorname{tr} A_{m-\nu}^{-k}}{\operatorname{tr} A_\nu^k \operatorname{tr} A_{m-\nu}^{-(k+1)}} = \operatorname{Det} A.$$

As a generalization of Theorem 6., we obtain on the basis of Theorem 3. and of Lemma 3. the following

Theorem 12. *If, besides the conditions of Theorem 3.,*

$$\lambda_1 \geqq \cdots \geqq \lambda_r \geqq 0, \quad \lambda_1 > 0,$$

then

$$\frac{\operatorname{Per}(A\Lambda^{k+1}B^T)}{\operatorname{Per}(A\Lambda^k B^T)} \uparrow \lambda_1^m, \qquad k \to \infty,$$

and

$$\sqrt[k]{\operatorname{Per}(A\Lambda^k B^T)} \uparrow \lambda_1^m, \qquad k \to \infty.$$

References

[1] Гнеденко, Б. В.—Колмогоров, Предельные распеделения дця сумм независимых случайных величин. *Москва—Ленинграб*. 1949.
[2] M. Marcus and M. Newman, Permanents functions. *Ann. Math.* **75** (1962), 45—62.
[3] M. Marcus and M. Minc, Permanents. *Monthly 72* (1965), 577—591.
[4] R. v. Mises, Wahrscheinlichkeitsrechnung. *Leipzig—Wien*, 1931.
[5] O. Perron, Algebra II. *Berlin—Leipzig*, 1927.
[6] B. L. van der Waerden, Aufgabe 45. *Jber.-Deutsch. Math. Verein.* **35** (1926), 117.

(Received April 3, 1972.)