On generalised Stieltjes transforms

By. J. L. ARORA (Pilani)

1. *Introduction*

If the integral

$$(1.1) \qquad \varphi(p) = \int_0^\infty (x+p)^{-1} f(x)\, dx$$

exists, then $\varphi(p)$ is called the *Stieltjes transform* of $f(x)$. $\varphi(p)$ is called the image of $f(x)$ under the kernel $(x+p)^{-1}$.

If the integral

$$(1.2) \qquad \varphi(p) = \int_0^\infty (x+p)^{-\lambda} f(x)\, dx$$

exists, then $\varphi(p)$ is called the *generalised Stieltjes transform* of $f(x)$ of order λ. Generalised Stieltjes transform of $f(x)$ of different orders are connected with each other by fractional integration. A generalisation of the aforesaid transform is given by VARMA [1]. ARYA [2], JOSHI [3], SAKSENA [4] and others have also worked on the generalisation given by Varma.

In the present note we take up a new generalisation of the above transform. We give Uniqueness theorem, Inversion formula and other theorems connected with our generalisation.

2. We consider the integral

$$(2.1) \qquad F(p) = \int_0^\infty (x^m+p^m)^{-\lambda} f(x)\, dx, \qquad \lambda, m>0,$$

provided of course, the integral on the right exists. Here also we call $F(p)$ the image of $f(x)$ under the kernel $(x^m+p^m)^{-\lambda}$. The integration in (2. 1) is over the positive real axis and we take p a positive real number. We write (2. 1) symbolically as

$$(2.2) \qquad F(p) \underset{m}{\overset{\lambda}{\to}} f(x), \qquad \lambda, m, p>0.$$

(2. 1) reduces to (1. 2) if we take $m=1$ and to (1. 1) if we take $\lambda=m=1$.

3. We now present the Uniqueness theorem which is the direct consequence of a lemma given by Bose [5].

Lemma. Let $G(x)$ be continuous in $(0, X)$ and absolutely integrable in $(0, \infty)$ and

$$\int_0^\infty (x^m + p^m)^{-\lambda} G(x)\, dx = 0, \qquad p, m, \lambda > 0. \tag{3. 1}$$

Then

$$G(x) \equiv 0. \tag{3. 2}$$

Using the above lemma, the proof of Uniqueness theorem stated below follows at once.

Theorem 1. *Let $f(x)$ and $g(x)$ be continuous and absolutely integrable in $(0, \infty)$ and*

$$F(p) \underset{m}{\overset{\lambda}{\to}} f(x) \tag{3. 3}$$

and also

$$F(p) \underset{m}{\overset{\lambda}{\to}} g(x). \tag{3. 4}$$

Then

$$f(x) \equiv g(x). \tag{3. 5}$$

4. *An Inversion Formula*

Titchmarsh [6], Widder [7] gave inversion formulae for (1. 1).

If

$$F(p) \underset{1}{\overset{1}{\to}} f(x), \tag{4. 1}$$

Then

$$f(x) = \frac{i}{2\pi} [F(xe^{i\pi}) - F(xe^{-i\pi})] \tag{4. 2}$$

under the conditions stated therein.

Now we give an inversion formula for (2. 1).

Theorem 2. *Let $f(x)$ be bounded for $x \geqq 0$ and of bounded variation in the neighbourhood of x, and*

$$F(p) \underset{m}{\overset{\lambda}{\to}} f(x). \tag{4. 3}$$

Then

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{m\Gamma(\lambda) G(s) x^{-s+m\lambda-1}}{\Gamma\left(\frac{s}{m}\right) \Gamma\left(\lambda - \frac{s}{m}\right)} ds, \qquad c > 0, \tag{4. 4}$$

where $G(s)$ is the Mellin transform of $F(x)$, provided that $x^{c-m\lambda} f(x) \in L(0, \infty)$,

$x^{c-1}F(x)\in L(0,\infty)$ *and* $0<\mathrm{Re}(s)<m\lambda$. *Further let* $f(x)$ *be continuous in* $(0,\infty)$, *then* (4. 4) *becomes*

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{m\Gamma(\lambda)G(s)x^{-s+m\lambda-1}}{\Gamma\left(\frac{s}{m}\right)\Gamma\left(\lambda-\frac{s}{m}\right)}ds,\qquad c>0. \tag{4. 5}$$

Proof. Let $G(s)$ be the Mellin transform of $F(x)$. Then

$$G(s)=\int_0^\infty p^{s-1}F(p)\,dp=\int_0^\infty p^{s-1}\int_0^\infty(x^m+p^m)^{-\lambda}f(x)\,dx\,dp=$$

$$=\int_0^\infty f(x)\int_0^\infty p^{s-1}(x^m+p^m)^{-\lambda}\,dp\,dx.$$

The change in the order of integration is justified under the conditions stated above. Now on evaluating the p-integral, we get

$$\frac{m\Gamma(\lambda)G(s)}{\Gamma\left(\frac{s}{m}\right)\Gamma\left(\lambda-\frac{s}{m}\right)}=\int_0^\infty x^{s-m\lambda}f(x)\,dx. \tag{4. 6}$$

On applying the Mellin's Inversion formula, we get

$$\frac{1}{2}[f(x+0)+f(x-0)]=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{m\Gamma(\lambda)G(s)x^{-s+m\lambda-1}}{\Gamma\left(\frac{s}{m}\right)\Gamma\left(\lambda-\frac{s}{m}\right)}ds,\qquad c>0, \tag{4. 7}$$

and if $f(x)$ is continuous in $(0,\infty)$ then we get (4. 5). Hence the theorem.

Cor. 2. 1 If we take $m=1$, then (4. 5) reduces to

$$f(x)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(\lambda)G(s)x^{-s+\lambda-1}}{\Gamma(s)\Gamma(\lambda-s)}ds,\qquad c>0. \tag{4. 8}$$

This is a complex inversion formula for (1. 2).

Cor. 2. 2 If we take $\lambda=m=1$, then (4. 5) reduces to

$$f(x)=\frac{1}{2\pi i}[F(xe^{-i\pi})-F(xe^{i\pi})]$$

which is (4. 2).

5. In Laplace transform Goldstein [8] proved the following theorem:

If

$$F(p)\doteq f(t) \tag{5. 1}$$

$$G(p)\doteq g(t), \tag{5. 2}$$

then

$$\int_0^\infty F(p)g(p)\frac{dp}{p} = \int_0^\infty G(t)f(t)\frac{dt}{t} \tag{5. 3}$$

provided the change in the order of integrations involved is justified. Here we have an analogue of this theorem.

Theorem 3. *Let*

$$\text{(i)} \quad F(p) \underset{m}{\overset{\lambda}{\to}} f(x) \tag{5. 4}$$

$$\text{(ii)} \quad G(p) \underset{m}{\overset{\lambda}{\to}} g(x). \tag{5. 5}$$

Then

$$\int_0^\infty F(t)g(t)\,dt = \int_0^\infty G(t)f(t)\,dt, \tag{5. 6}$$

provided the changes in the order of integrations involved are justified.

The proof of this theorem is simple. It is similar to that of Goldstein theorem. Hence we omit the proof.

Theorem 4. *Let*

$$\text{(i)} \quad F(p) \underset{m}{\overset{\lambda}{\to}} f(x) \tag{5. 7}$$

$$\text{(ii)} \quad G(p) \underset{m}{\overset{\lambda}{\to}} g(x). \tag{5. 8}$$

Then

$$p^{1-m\lambda}\int_0^\infty G(t)f(pt)\,dt \underset{m}{\overset{\lambda}{\to}} \int_0^\infty t^{1-m\lambda}\,g(t)f(xt)\,dt \tag{5. 9}$$

provided that the various changes in the order of integrations are justified.

Proof. We have

$$F(p) = \int_0^\infty (x^m + p^m)^{-\lambda} f(x)\,dx.$$

Now on substituting $x = ax$ and $p = ap$, we get

$$a^{m\lambda-1}\,F(ap) \underset{m}{\overset{\lambda}{\to}} f(ax). \tag{5. 10}$$

Now on applying theorem 3 to (5. 8) and (5. 10), we get

$$a^{1-m\lambda}\int_0^\infty G(t)f(at)\,dt = \int_0^\infty F(at)g(t)\,dt = \int_0^\infty g(t)\int_0^\infty (x^m + a^m t^m)^{-\lambda} f(x)\,dx\,dt.$$

Now on substituting $x=tx$, replacing a by p and changing the order of integrations, provided it is permissible, we obtain

$$p^{1-m\lambda}\int_0^\infty G(t)f(pt)\,dt = \int_0^\infty (x^m+p^m)^{-\lambda}\int_0^\infty t^{1-m\lambda}\,g(t)f(xt)\,dt\,dx$$

$$\frac{\lambda}{\dot{m}}\int_0^\infty t^{1-m\lambda}\,g(t)f(xt)\,dt.$$

Hence the theorem.

Cor. 4. 1
Let

$$F(p)\,\frac{1}{\dot{1}}\,f(x).$$

Then

(5. 11)
$$\int_0^\infty F(t)f(pt)\,dt\,\frac{1}{\dot{1}}\int_0^\infty f(t)f(xt)\,dt.$$

The above corollary can be easily established by taking $\lambda=m=1$ and $g(t)=f(t)$.

6. Consider the integral equation [9]

(6. 1)
$$k(x) = \int_0^\infty f(t)f(xt)\,dt$$

were $k(x)$ is a known function.

1. (6. 1) has a solution only if $xk(x)=k\left(\frac{1}{x}\right)$ and the integral on the right exists.
2. Let $k(x)\in L^2(0,\infty)$ and $K(s)$ be the Mellin transform of $f(x)$. Then $K(s) = K(1-s)$. Further, let $K(s) = H(s)H(1-s)$, $H(s)\in L^2(\frac{1}{2}-i\infty, \frac{1}{2}+i\infty)$. Then $h(x)$, the inverse Mellin transform of $H(s)$ is a solution of (6. 1).
3. If $f(x)\in L^2(0,\infty)$ and is a solution of (6. 1), then image of $f(x)$ under any Fourier kernel is also a solution of (6. 1).

Now consider the integral

(6. 2)
$$I = \int_0^\infty M(pt)\,M(t)\,dt,$$

where

(6. 3)
$$pM(p) \doteq f(t).$$

$$\therefore \# I = \int_0^\infty M(t)\,dt\int_0^\infty e^{-pty}f(y)\,dy = \int_0^\infty f(y)\,dy\int_0^\infty e^{-pty}\,M(t)\,dt =$$

$$= \int_0^\infty f(y)\,dy\int_0^\infty e^{-pty}\,dt\int_0^\infty e^{-tx}f(x)\,dx.$$

The change in the order of integration is justified provided $M(t)\in L(0,\infty)$

and $f(x) \in L^2(0, \infty)$. Now on changing the order of t and x integrals, which is justified, we get

$$I = \int_0^\infty f(y)\, dy \int_0^\infty (x+y)^{-1} f(px)\, dx = \int_0^\infty f(px)\, dx \int_0^\infty (x+y)^{-1} f(y)\, dy.$$

The change in the order of integration is justified. Hence we get

$$I = \int_0^\infty f(px) F(x)\, dx, \tag{6. 4}$$

where

$$F(p) \underset{1}{\overset{1}{\rightarrow}} f(x).$$

Now we state a theorem which directly follows from (5. 11) and (6. 4).

Theorem 5. *Let* (i) $f(t)$ *be a solution of the integral equation*

$$k(x) = \int_0^\infty f(xt) f(t)\, dt, \tag{6. 5}$$

$$\text{(ii)} \quad pM(p) \doteqdot f(t). \tag{6. 6}$$

Then

$$\int_0^\infty M(pt) M(t)\, dt \underset{1}{\overset{1}{\rightarrow}} k(x) \tag{6. 7}$$

provided the conditions stated above are satisfied. If $M(t) \in L^2(0, \infty)$, *then* $M(t)$ *can be replaced by its image under any Fourier kernel.*

Example. Let $k(x) = (x+1)^{-1}$. Then [10]

$$K(s) = \Gamma(s)\Gamma(1-s), \quad H(s) = \Gamma(s), \quad h(x) = e^{-x}.$$

$$\therefore \# f(t) = e^{-t}, \quad M(t) = (t+1)^{-1}.$$

Hence, from (6. 7), we get

$$\frac{1}{p-1} \log p \underset{1}{\overset{1}{\rightarrow}} \frac{1}{1+x} \tag{6. 8}$$

which is a known result. If $f_c(x)$, $f_s(x)$, and $f_\mu(x)$ denote the Fourier cosine, Fourier sine and Hankel transform of $f(x)$, respectively, then

$$f_c(x) = \sqrt{(2/\pi)}\, (1+x^2)^{-1}; \quad f_s(x) = \sqrt{(2/\pi)}\, \frac{x}{x^2+1};$$

$$f_\mu(x) = \frac{2^{-\mu} x^{\mu+(1/2)} \Gamma\left(\mu + \frac{3}{2}\right)}{\Gamma(\mu+1)}\, {}_2F_1\left(\frac{2\mu+3}{4}, \frac{2\mu+5}{4}; \mu+1; -x^2\right), \qquad \operatorname{Re}(\mu) > -\frac{3}{2}.$$

Obviously $f(x)$, $f_c(x)$, $f_s(x)$, $f_\mu(x)$ are solutions of the equation (6. 5). Thus we have

$$(6.8)\qquad \frac{1}{1+x}=\frac{2}{\pi}\int_0^\infty \frac{dt}{(1+t^2)(1+x^2t^2)}=\frac{2x}{\pi}\int_0^\infty \frac{t^2\,dt}{(1+t^2)(1+x^2t^2)}=$$

$$=\frac{\left[\Gamma\left(\mu+\frac{3}{2}\right)\right]^2 x^{\mu+(1/2)}}{2^{2\mu}[\Gamma(\mu+1)]^2}\int_0^\infty t^{2\mu+1}\,{}_2F_1\left[\begin{matrix}\frac{\mu}{2}+1\pm\frac{1}{4};\ -t^2\\ \mu+1;\end{matrix}\right]\times$$

$$\times\,{}_2F_1\left[\begin{matrix}\frac{\mu}{2}+1\pm\frac{1}{4};\ -x^2t^2\\ \mu+1;\end{matrix}\right]dt,\qquad \mathrm{Re}(\mu)>-\frac{3}{2}.$$

Since $M(t)$ can also be replaced by its any Fourier image, we have

$$(6.9)\qquad \frac{\pi}{2}\cdot\frac{\log p}{p-1}=\frac{\pi}{2}\int_0^\infty \frac{dt}{(1+t)(1+pt)}=$$

$$=\int_0^\infty [\mathrm{si}(pt)\sin(pt)+\mathrm{Ci}(pt)\cos(pt)][\mathrm{si}(t)\sin(t)+\mathrm{Ci}(t)\cos(t)]\,dt=$$

$$=\int_0^\infty [\mathrm{Ci}(pt)\sin(pt)-\mathrm{si}(pt)\cos(pt)][\mathrm{Ci}(t)\sin(t)-\mathrm{si}(t)\cos(t)]\,dt.$$

7. Now on substituting $f(t)=t^{n+(1/2)}J_\mu(t)$ and $t^n e^{-t}$ in theorem 4, we get the following two theorems:

Theorem 6. *Let*

(7. 1) (i) $F(p)\ \frac{\lambda}{\overrightarrow{m}}\ f(x)$,

(7. 2) (ii) *$H(p)$ be the Hankel transform of order μ of $t^n F(t)$,*

(7. 3) (iii) *$G(x)$ be the Hankel transform of order μ of $t^{1+n-m\lambda}f(t)$.*

Then

$$(7.4)\qquad p^{1+n-m\lambda}H(p)\ \frac{\lambda}{\overrightarrow{m}}\ x^n G(x),$$

provided $t^n F(t)$ and $t^{1+n-m\lambda}f(t)$ are continuous and absolutely integrable in $(0,\infty)$ and $x^n G(x)=0(x^\alpha)$, $\alpha>-1$ for small x and $x^{n-m\lambda}G(x)=0(x^{-1-\delta})$, $\delta>0$ for large x.

Theorem 7. *Let*

$$\text{(7.5)}\qquad \text{(i)}\quad F(p)\ \underset{m}{\overset{\lambda}{\rightarrow}}\ f(x),$$

$$\text{(7.6)}\qquad \text{(ii)}\quad H(p) \doteqdot t^{n} F(t),$$

$$\text{(7.7)}\qquad \text{(iii)}\quad G(p) \doteqdot t^{1+n-m\lambda} f(t).$$

Then

$$\text{(7.8)}\qquad p^{n-m\lambda} H(p)\ \underset{m}{\overset{\lambda}{\rightarrow}}\ x^{n-1} G(x),$$

provided $H(p), G(p)$ *exist and* $x^{n-1} G(x)=0(x^{\alpha}), \alpha>-1$ *for small* x *and* $x^{n-1-m\lambda} G(x)= =O(x^{-1-\delta}),\ \delta>0$ *for large* x.

Now we shall give here a few examples which we believe to be new.

Examples on Theorem 6.

1. Let $\lambda=m=1,\ n=-\mu-\dfrac{1}{2}$ and $f(t)=t^{\mu} J_{\mu}(t)$. Then [10]

$$\text{(7.9)}\qquad \frac{\Gamma(\mu+1)}{2\sqrt{\pi}\,\Gamma\left(\mu+\frac{1}{2}\right)} G_{3;\,3}^{2;\,3}\left(p^{2}\,\middle|\,\begin{matrix}0,\ \frac{1}{2},\ \frac{1}{2}-\mu\\ 0,\ 0,\ -\mu\end{matrix}\right)\ \underset{1}{\overset{1}{\rightarrow}}\ {}_{2}F_{1}\left[\begin{matrix}\mu+\frac{1}{2},\ \frac{1}{2};\ x^{2}\\ \mu+1;\end{matrix}\right],$$

$$\operatorname{Re}(\mu)>-\frac{1}{2}.$$

2. Let $m=2,\ n=-\mu-\varrho$ and $f(t)=t^{2\varrho+1} e^{-t^{2}/4}$. Then [10]

$$\text{(7.10)}\qquad \frac{\Gamma(\mu+1) p^{-\lambda-\varrho}}{2\Gamma(\lambda)\,\Gamma\left(\frac{7}{4}+\frac{\varrho}{2}-\lambda\right)} G_{2;\,3}^{2;\,2}\left(p^{2}\,\middle|\,\begin{matrix}\frac{\lambda-\varrho}{2},\ \frac{2+\varrho-\lambda}{2}\\ \frac{3-2\lambda}{4},\ \frac{\lambda+\varrho}{2},\ \frac{3-2\lambda-4\mu}{4}\end{matrix}\right)\ \underset{2}{\overset{\lambda}{\rightarrow}}\ x^{(1/2)-\varrho}\times$$

$$\times\,{}_{1}F_{1}\left(\frac{2\varrho-4\lambda+7}{4};\ \mu+1;\ -x^{2}\right),$$

$$-1<\operatorname{Re}(\varrho)<\frac{3}{2},\quad \operatorname{Re}(\lambda)>0.$$

If $\varrho=2\lambda+2\mu-\dfrac{3}{2}$, then (7.10) reduces to a known result.

Examples on Theorem 7.

1. Let $\lambda=m=1$ and $f(t)=\sin(2\sqrt{2t})$. Then [10]

(7.11)

$$\sqrt{\pi}\,2^{3/2}\Gamma(2n+2)\cos(n\pi)e^{p}D_{-2n-2}(2\sqrt{p})\;\frac{1}{1}\;e^{-x}\left[D_{2n+1}(2\sqrt{x})-D_{2n+1}(-2\sqrt{x})\right],\quad n>0.$$

2. Let $\lambda=m=1$, $n=n-1$ and $f(t)=J_{2\mu}(2\sqrt{t})$. Then [10]

$$(7.12)\qquad \frac{\Gamma(\mu)\Gamma(2\mu+1)}{\Gamma(n+\mu)}\left[\frac{\Gamma(n)}{\Gamma(\mu+1)}\,{}_2F_2\begin{bmatrix}1,\ n;\ p\\ 1\pm\mu;\end{bmatrix}\right.$$

$$\left.-\frac{\Gamma(1-\mu)\Gamma(n+\mu)}{\Gamma(1+2\mu)\sqrt{p}}\,e^{p/2}M_{(1/2)-n,\mu}(p)\right]\;\frac{1}{1}\;x^{\mu}\,{}_1F_1(n+\mu;\ 2\mu+1;\ -x),$$

$$n>0,\quad \operatorname{Re}(\mu)>-1,\quad \operatorname{Re}(n+\mu)>0.$$

From (7.10) and (7.12), we easily get

$$(7.13)\qquad G^{2,2}_{2,3}\left(p^2\,\middle|\,\begin{matrix}\frac{1}{2}+2\mu,\ 0\\ \mu,\ 0,\ -\mu\end{matrix}\right)=\frac{1}{\mu}\Gamma\left(\frac{1}{2}-2\mu\right){}_2F_2\begin{bmatrix}1,\ \frac{1}{2}-2\mu;\ p^2\\ 1\pm\mu;\end{bmatrix}-$$

$$-\frac{\sqrt{\pi}\,2^{2\mu}\Gamma(\mu)\Gamma(1-2\mu)}{p\Gamma(1+2\mu)}\,e^{p^2/2}M_{2\mu,\mu}(p^2).$$

3. Let $\lambda=1$, $m=2$, $n=-1/2$ and $f(t)=tJ_{\mu}(t/2)J_{-\mu}(t/2)$. Then [10]

$$(7.14)\quad \frac{1}{2\pi}\sqrt{\frac{p}{\pi}}\,G^{2,4}_{4,4}\left(p^2\,\middle|\,\begin{matrix}0,\ \frac{1}{2},\ -\frac{1}{4},\ \frac{1}{4}\\ -\frac{1}{4},\ -\frac{1}{4},\ \mu-\frac{1}{4},\ -\mu-\frac{1}{4}\end{matrix}\right)\;\frac{1}{2}\;xP^{\mu}_{-1/4}(y)P^{-\mu}_{-1/4}(y),$$

$$y=\sqrt{1+x^2},\quad |\operatorname{Re}(\mu)|<1.$$

4. Let $\lambda=1$, $m=2$, $n=-1/2$ and $f(t)=t[J_{\mu}(t/2)]^2$. Then [10]

$$(7.15)\quad \frac{\sqrt{p}\,2^{\mu-1}}{\pi\Gamma\left(2\mu+\frac{1}{2}\right)}\,G^{2,3}_{3,3}\left(p^2\,\middle|\,\begin{matrix}0,\ \frac{1}{2},\ \frac{1}{4}\\ -\frac{1}{4},\ \mu-\frac{1}{4},\ -\mu-\frac{1}{4}\end{matrix}\right)\;\frac{1}{2}\;x\left[P^{-\mu}_{-1/4}(\sqrt{1+x^2})\right]^2,$$

$$\operatorname{Re}(\mu)>-\frac{1}{4}.$$

5. Let $\lambda=1$, $m=2$, $n=1$ and $f(t) = t[J_\mu(t/2)]^2$. Then [10]

$$(7.16)\quad \frac{\Gamma(\mu+1)2^{2\mu-1}}{p\sqrt{\pi}\,\Gamma\left(\mu+\frac{3}{2}\right)} G_{3,3}^{2,3}\left(p^2\left|\begin{matrix}0,\ \frac{1}{2},\ 1\\ \frac{1}{2},\ \frac{1}{2}+\mu,\ \frac{1}{2}-\mu\end{matrix}\right.\right) \frac{1}{2}\, x^{2\mu+1}\,{}_2F_1\left[\begin{matrix}\mu+1\pm\frac{1}{2};\ -x^2\\ 2\mu+1;\end{matrix}\right],$$

$$\operatorname{Re}(\mu) > -1.$$

In the end, I wish to express my sincere thanks to Dr. S. C. Mitra for his keen interest in the preperation of this paper.

References

[1] R. S. Varma, On a generalisation of Laplace integral, *Proc. Nat. Aca. Sc., India,* **A20** (1951), 209—216.

[2] Suresh Chandra Arya, Some theorems connected with a generalised stieltjes transform, *Bull. Cal. Math. Soc.* **51** (1) (1959), 39—47.

[3] J. M. C. Joshi, On a generalised stieltjes transform, *Pacific J. Math.* **14** (3) (1964), 969—975.

[4] K. M. Saksena, Generalisation of Stieltjes transform, *Bull. Cal. Math. Soc.* **45** (3) (1953), 101—107.

[5] B. N. Bose, Certain theorems on self-recpirocal functions, *Bull. Cal. Math. Soc.* **46** (1954), 109—127.

[6] E. C. Titchmarsch, Theory of Fourier Integrals, *Oxford,* 1948.

[7] D. V. Widder, The Laplace Transform. *New York,* (1941).

[8] S. Goldestein, Operational representation of Whittaker's Confluent hypergeometric functions and Weber's parabolic cylinder function. *Proc. Lond. Math. Soc.* (2), **34** (1932), 103—125.

[9] S. C. Mitra, Theory of Transforms since 1948, BITS, *Pilani* (1969).

]10] A. Erdélyi, Tables of Integral Transforms, Vol I and II, *New York* (1953).

(Received March 19, 1971)