# An integrability theorem for power series

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**1.** HEYWOOD [2] proved the following result:

**Theorem A.** *Suppose that* $F(x)=\sum_0^\infty c_n x^n$ *for* $0 \leqq x<1$, $\gamma<1$ *and that there is a positive number* $\varepsilon$ *such that* $c_n > \frac{-K}{n^{\gamma+\varepsilon}}$, *for all sufficiently large values of* $n$. *(*$K$ *some positive constant.) Then* $(1-x)^{-\gamma} F(x) \in L(0,1)$ *iff* $\sum n^{\gamma-1}|c_n|$ *converges.*

The object of this present note is to obtain a generalization of Theorem A.

**2.** We prove the following result:

**Theorem.** *Let*

$$F(x)=\sum_0^\infty c_n x^n, \qquad 0 \leqq x<1 \text{ and } \gamma<1.$$

*Suppose that there is a positive number* $\varepsilon$ *such that*

$$c_n > \frac{-K}{n^{(\gamma/p)+(1-1/p)+\varepsilon}} \qquad (0<p \leqq \infty,\ K \text{ constant}) \tag{2.1}$$

*for all sufficiently large values of* $n$. *Then* $(1-x)^{-\gamma}(F(x))^p \in L(0,1)$ *iff*

$$\sum n^{\gamma-2}\left(\sum_{k=1}^n |c_k|\right)^p$$

*converges.*

It may be remarked that if we put $p=1$ in our theorem, we get Theorem A.

**3.** We require the following lemmas.

**Lemma 1.** [5, p. 58]. *If* $b$ *is a constant, then*

$$\frac{\Gamma(x)}{\Gamma(x+b)} \sim x^{-b}, \quad \text{as} \quad x \rightarrow \infty.$$

**Lemma 2.** [3]. *Let*

$$F(x) = \sum_0^\infty c_n x^n, \quad c_n \geqq 0, \ 0 \leqq x < 1, \qquad s_n = \sum_{k=1}^n c_k,$$

*and* $\gamma < 1$. *Then, for* $0 < p \leqq \infty$,

$$\Big(\int_0^1 (1-x)^{-\gamma}(F(x))^p\,dx\Big)^{1/p} < \infty \quad \textit{iff} \quad \left(\sum_1^\infty n^{\gamma-2} s_n^p\right)^{1/p} < \infty.$$

**Lemma 3.** [1, p. 255]. *If* $c > 1$,

$$s_n = \sum_{k=1}^n a_k, \qquad a_k \geqq 0,$$

*then* $\sum n^{-c} s_n^p \leqq K \sum n^{-c}(na_n)^p$ $(p \geqq 1)$.

**Lemma 4.** [4, p. 83]. *If* $c > 1$, $0 < p < 1$, $a_n \geqq 0$ *and* $\{n^{-j} a_n\}$ *is monotonic decreasing for some* $j > 0$, *then*

$$\sum n^{-c}\left(\sum_1^n a_k\right)^p \leqq K \sum n^{-c}(na_n)^p.$$

**4.** Proof of the Theorem. We may suppose without loss that $\dfrac{\gamma}{p} - \dfrac{1}{p} + \varepsilon$ is not an integer. This will ensure the existence of the following gamma function at all relevant points. Let

$$G(x) \equiv 2K\Gamma\left(\frac{1-\gamma}{p} - \varepsilon\right)(1-x)^{(\gamma-1)/p+\varepsilon} = \sum_0^\infty a_n x^n, \quad \text{for} \quad 0 \leqq x < 1.$$

Then, since

$$(1-x)^{(\gamma-1)/p+\varepsilon} = \sum_0^\infty \frac{\Gamma\left(n + \dfrac{1-\gamma}{p} - \varepsilon\right)}{\Gamma(n+1)\,\Gamma\left(\dfrac{1-\gamma}{p} - \varepsilon\right)} x^n,$$

we have

$$a_n = 2K \frac{\Gamma\left(n + \dfrac{1-\gamma}{p} - \varepsilon\right)}{\Gamma(n+1)} \sim \frac{2K}{n^{(\gamma-1)/p+1+\varepsilon}}, \quad \text{as} \quad n \to \infty, \tag{4.1}$$

by Lemma 1. It follows from (2. 1) that $c_n + a_n$ is positive for all sufficiently large values of $n$. Since

$$F(x) + G(x) = \sum_0^\infty (c_n + a_n) x^n$$

for $0 \leqq x < 1$. Lemma 2 now shows that $(1-x)^{-\gamma}(F(x)+G(x))^p \in L(0, 1)$ iff

$$\sum n^{\gamma-2}\left(\sum_{k=1}^n (a_k + c_k)\right)^p$$

converges.

But $(1-x)^{-\gamma}G^p(x)$ is a multiple of $(1-x)^{\varepsilon p-1}$ and is therefore integrable $L$ in $(0, 1)$. Moreover (4. 1) shows that $\sum n^{\gamma-2}\left(\sum_{1}^{n} a_k\right)^p$ is convergent by Lemmas 3 and 4. Therefore it follows that $(1-x)^{-\gamma}(F(x))^p \in L(0, 1)$ iff

$$\sum n^{\gamma-2}\left(\sum_{1}^{n} |c_k|\right)^p < \infty$$

Thus the theorem is proved.

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## References

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