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A note on Tamássy's paper in view of Miron spaces

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Abstract. Analytic-geometric relations between the Miron spaces (often called Cartan spaces) and generalized Miron spaces having the same base manifold are sketched. It is shown that the two can be related by a second order envelope of a particular surface.

A Finsler space $F^n = (M, L)$ over a manifold M is given by the fundamental function $L : TM \to R$, in local coordinates by $L(x, y)$, $x \in M$, $y \in T_x M$ where L is homogeneous of second order in y. Then the metric tensor $g_{ij}(x, y) := \frac{1}{2} \frac{\partial}{\partial y^i} \frac{\partial L}{\partial y^j}$ is derived from L. (The fundamental function is often considered as $\sqrt{L} \equiv \mathcal{L}$ and then $g_{ij} = \frac{1}{2} \frac{\partial}{\partial y^i} \frac{\partial \mathcal{L}^2}{\partial y^j}$). However, instead of L one can start with an apriori given $g_{ij}(x, y)$. The space (M, g_{ij}) is called a generalized Finsler space. L. TAMÁSSY [3] recently gave analytic geometric relations between these two spaces.

Following the duality between the Miron spaces M^n [1] (often called also Cartan spaces [2]) and the Finsler spaces F^n one sees that the differential geometry of M^n has the same importance as the geometry of F^n. Further there exists a generalization of the M^n similar to that of the F^n mentioned above, called generalized Miron (Cartan) space. In this short note, considering Tamássy's paper [3], we point out the similar analytic-geometric relations between the Miron and generalized Miron spaces having the same base manifold.

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1. Introduction

A Miron space is a pair $M^n = (M, H)$ where $H : T^*M \to R$ is a function having the properties: (i) $H(x,p)$ is 2-homogeneous with respect to p_i, (ii) its Hessian $g^{ij}(x,p) = (\dot{\partial}^i\dot{\partial}^j H)/2$ satisfies $\det(g^{ij}) \neq 0$, for every $(x,p) \in T^*M\backslash\{0\}$. Here $\zeta^n = (T^*M, \pi^*, M)$ denotes the cotangent bundle of the n-dimensional C^∞-manifold M and $x = (x^i)$ $(i, j, k, \ldots = \overline{1, n})$ is a coordinate system in $U \subset M$. It induces a canonical coordinate system $(x,p) = (x^i, p_i)$ in T^*M by $p^i = p(dx^i)$. The Hamilton function H is called the fundamental function of M^n and the d-tensor field g^{ij} ([2]) is called the metric fundamental tensor of M^n. In the following H is supposed to be positive. Further we consider a Miron space over M built directly on the metric tensor g^{ij} and denote it by (M, g^{ij}). Here g^{ij} has the following properties.

(i) It is a d-tensor field of type (2,0) symmetric and non-degenerate.

(ii) It is o-homogeneous with respect to p_i.

We restrict ourselves only to the case when g^{ij} is positive definite.

The purpose of the paper is to discuss the analytic geometric relations between the spaces (M, H) and (M, g^{ij}).

2. Analytic geometric relations between the spaces (M, H) and (M, g^{ij})

We consider a quadric

$$g^{ij}(x_0, p_0)p_ip_j = 1,$$

in $T^*_{x_0}M$ (the space of all covectors p_i at x_0) for a fixed point (x_0, p_0). Then

$$\mathcal{F}(p, c) : g^{ij}(x_0, c)p_ip_j - 1 = 0 \tag{2.1}$$

is a family of quadrics depending on the parameters $c \in \mathbb{R}^n$ $(c \neq 0)$. Further

$$F(c) : p_i = p_i(c) := \frac{c_i}{\left(g^{\ell m}(x_0, c)c_\ell c_m\right)^{1/2}} \tag{2.2}$$

is a surface $F \subset T^*_{x_0}M$ in a parametrised form.

We call $F(c)$ an *envelope of* $\mathcal{F}(p, c)$ if (i) the point $P(c)$ determined by (2.2) lies on the surface (2.1), and (ii) in these points they touch each other. Further, we call F a second order envelope of $\mathcal{F}$ if it is an envelope and $\mathcal{F}(p,c)$ and F osculate at the common point $P(c)$ in the second order.

Then we obtain results analogous to those found by Tamássy in [3] for Finsler spaces. Since proofs in [3] are in a hardly accessible proceedings, it is reasonable to publish the proofs of the following propositions and the theorem.

We now prove

Proposition 2.1. *F is an envelope of $\mathcal{F}(p,c)$ if and only if*

$$C^{ijk}p_ip_j = 0, \tag{2.3}$$

where $C^{ijk} = -\frac{1}{2}\dot{\partial}^k g^{ij}$ is a well-known d-tensor (cf. [2]).

Proof. Obviously

$$\mathcal{F}(p(c),c) = 0. \tag{2.4}$$

By differentiation it gives

$$\frac{\partial \mathcal{F}}{\partial p_h}(p(c),c)\frac{\partial p_h}{\partial c_s} + \frac{\partial \mathcal{F}}{\partial c_s}(p(c),c) = 0. \tag{2.5}$$

Endowing $T^*_{x_0}M$ with a euclidean metric such that p_h are euclidean coordinates

$$\dot{\partial}^h \mathcal{F}(p(c),c) = 2g^{hi}(x_0,c)p_i(c), \quad \dot{\partial}^h \equiv \frac{\partial}{\partial p_h} \tag{2.6}$$

are the components of the normal vector, say $N_{\mathcal{F}}$ of $\mathcal{F}(p,c)$ at the point $P(c)$. It is remarked that $N_{\mathcal{F}} \neq 0$, otherwise $p_i(c) = 0$ in view of (2.6), which is a contradiction due to (2.2) and $c \neq 0$. Further $\frac{\partial p_h(c)}{\partial c_s}$ are the components of tangents, say $T^*_{(s)}F$ that span the tangent space of $F(c)$. Then the first term on the left hand side of (2.5) vanishes as the product of a tangent and a normal vector in a euclidean space. So under the assumption that F is an envelope of $\mathcal{F}(p,c)$, we find

$$\frac{\partial \mathcal{F}(p(c),c)}{\partial c_s} = 0. \tag{2.7}$$

On the other hand, if (2.7) holds then the vectors $N_{\mathcal{F}}$ and $T^*_{(s)}F$ are perpendicular in the eucidean cotangent space $T^*_{x_0}M$. Therefore in view of (2.4) F is an envelope of $\mathcal{F}(p,c)$.

Moreover, in view of (2.1) and (2.2),

$$\left(\frac{\partial \mathcal{F}}{\partial c_k}(p,c)\right)_{p=p(c)}$$
$$= \frac{\partial g^{ij}(x_0,c)}{\partial c_k}\,\frac{c_i}{(g^{\ell m}(x_0,c)c_\ell c_m)^{1/2}}\,\frac{c_j}{(g^{uv}(x_0,c)c_u c_v)^{1/2}} = 0.$$

On multiplication by $g^{\ell m}(x_0,c)c_\ell c_m$ and denoting the second variable of g^{ij} by p, the result follows.

Proposition 2.2. *F is a second order envelope of $\mathcal{F}$ if and only if* (2.3), *and*

(2.8) $$C^{ijk}p_i = 0.$$

Proof. Let F be a second order envelope of $\mathcal{F}$.

We first note that the surface S of T_{x_0} defined by

$$g^{ij}(x_0,p)p_i p_j = 1$$

is the same as F, for $F \subset S$ in view of (2.2), and F and S have a single point on each ray in $T^*_{x_0}$ through x_0. Furthermore we have (2.3) due to our assumption. From (2.3) we find

(2.9) $$\left(\dot\partial^r \dot\partial^s g^{ij}\right) p_i p_j + 2\left(\dot\partial^r g^{is}\right) p_i = 0,$$

by differentiation.

Also the second order osculation of $\mathcal{F}$ and F implies the equality of

(2.10) $$\dot\partial^r \dot\partial^s S(x_0,p)$$
$$= \dot\partial^r \dot\partial^s g^{ij}(x_0,p)p_i p_j + 2\dot\partial^s g^{ir}(x_0,p)p_i + 2\dot\partial^r g^{is}(x_0,p)p_i + 2g^{rs}(x_0,p),$$

and

(2.11) $$\dot\partial^r \dot\partial^s \mathcal{F}(x_0,p) = 2g^{rs}(x_0,c)$$

at point $P(c)$. (Due to homogeneity of order 0 of g^{rs}, we note that $g^{rs}(x_0,p(c)) = g^{rs}(x_0,c)$.)

From (2.10) we find (2.8) in view of (2.9) and (2.11).

On the other hand, if (2.8) holds, then we have (2.3) i.e. F is an envelope of $\mathcal{F}$. Further from (2.10) we have

$$\dot\partial^r \dot\partial^s S(x_0,p) = 2g^{rs}(x_0,p),$$

due to (2.8) and (2.9) which follows from (2.3). Hence

$$\dot\partial^r\dot\partial^s S = \dot\partial^r\dot\partial^s \mathcal{F}$$

at $P(c)$, i.e. F and $\mathcal{F}$ osculate in second order at $P(c)$. Consequently, F is a second order envelope of $\mathcal{F}$.

Using Propositions (2.1) and (2.2) we have an analytic-geometric characterization of the relation between the spaces (M,H) and (M,g^{ij}), namely

Theorem. *The space (M,g^{ij}) is deduced from a Miron space (M,H) if and only if $F(c)$ is a second order envelope of $\mathcal{F}(p,c)$.*

Proof. From

$$g^{ij}(x,p) = (\dot\partial^i\dot\partial^j H)/2, \tag{2.12}$$

(H is 2-homogeneous with respect to p_i) we have

$$\begin{aligned}\dot\partial^k g^{ij}(x,p)p_i &= \left(\dot\partial^k\dot\partial^i\dot\partial^j H(x,p)\right)p_i/2\\ &= \left[\left(\dot\partial^i\dot\partial^k\dot\partial^j H(x,p)\right)p_i\right]/2 = 0\end{aligned}$$

(by Euler's theorem). Thus (2.8) and (2.3) are satisfied. Consequently F is a second order envelope of $\mathcal{F}$.

On the other hand, let F be a second order envelope of $\mathcal{F}$ and

$$H(x,p) := g^{ij}(x,p)p_ip_j.$$

Then

$$\dot\partial^k H(x,p) = \dot\partial^k g^{ij}(x,p)p_ip_j + 2g^{kj}(x,p)p_j$$

yields

$$\dot\partial^k H(x,p) = 2g^{kj}(x,p)p_j$$

due to Proposition 2.1, and further by differentiation

$$\dot\partial^h\dot\partial^k H(x,p) = 2(\dot\partial^h g^{kj}(x,p)p_j + 2g^{kh}(x,p)) = 2g^{kh}(x,p)$$

due to Proposition 2.2.

This completes the theorem.

Proposition 2.3. *g^{ij} can be deduced from an H by (2.12), if and only if*

$$\left(\dot\partial^k g^{ij}\right)p_i = 0.$$

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References

[1] Gh. Atanasiu and M. Hasiguchi, Semi-symmetric Miron connections in dual Finsler spaces, *Rep. Fac. Sci. Kagoshima Univ. (Math. Phys. Chem.)* **20** (1987), 43–49.

[2] R. Miron, Hamilton geometry, *Seminarul de Mecanicá* Nr. **3**, *Univ. Timişoara*, 1987.

[3] L. Tamássy, On the existence of a fundamental function in Finsler spaces, Proc. Sixth. Nat. Sem. on Finsler, Lagrange and Hamilton spaces, Univ. Braşov, 1990, pp. 157–164.

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