A basis theorem for the semiring part of a Boolean algebra

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In this paper a study of a class of semirings associated with a Boolean algebra will be given. For such semirings an analog of the following classical theorem of Hilbert will be developed:

Hilbert Basis Theorem: *If R is a ring with identity satisfying the ascending chain condition on ideals, then $R[x]$, the ring of polynomials in a transcendental element over R, also satisfies the ascending chain condition on ideals.*

Throughout this paper, the binary operations of a Boolean algebra will be denoted by $+$ and $\cdot$ in place of the more common $\cup$ and $\cap$. Also apostrophe $'$ will be used to denote the unary operation.

Definition 1: Let $(R, +, \cdot, ')$ be a Boolean algebra. Then $(R, +, \cdot)$ is a commutative semiring with zero and identity. Such a semiring will be called a B-semiring.

The following theorem is a well known result in the theory of Boolean algebras although it has been slightly rephrased to fit the development presented here.

Theorem 2. *Let $(R, +, \cdot)$ be a B-semiring. Define the binary operation "$\oplus$" on R by $x\oplus y = xy'+x'y$. Then, $R^*=(R, \oplus, \cdot)$ is a ring, and I is an ideal in the B-semiring R if and only if I is an ideal in the ring R^*.*

Corollary 3: Let $(R, +, \cdot)$ be a B-semiring. Then R satisfies the ascending chain condition on ideals if and only if R^* satisfies the ascending chain condition on ideals.

Let $R[x]$ be the semiring of polynomials in a transcendental element x over the B-semiring $(R, +, \cdot)$. By inserting sufficient zero's, it is clear that all polynomials in $R[x]$ may be assumed to have the same number of terms. The unary operation on R may now be used to define an analog of subtraction on $R[x]$.

Definition 4: Suppose $f(x)=\sum a_i x^i$, $g(x)=\sum b_i x^i$ are in $R[x]$. Define the binary operation "$-$" on $R[x]$ by $f(x)-g(x) = \sum a_i b_i' x^i$.

Those ideals that are closed under $-$ will be the principal objects of study in this paper.

Definition 5: An ideal I of $R[x]$ is complemented if $f(x)$, $g(x)\in I$ imply $f(x)-g(x)\in I$.

Example 6: If R is any B-semiring, let I be the set of all polynomials in $R[x]$ having constant term equal zero. Then I is a complemented ideal.

Example 7: If I is an ideal in R, then $I[x]=\{f(x)\in R[x] \mid$ the coefficients of $f(x)$ are in $I\}$ is a complemented ideal.

It will be shown that Example 6 is a special case of a more general situation. A construction process will be presented by which complemented ideals may be constructed in $R[x]$ for any B-semiring R.

Lemma 8. *If $f(x), g(x)\in R[x]$ and $c\in R$, then $f(x)-g(x)\cdot cx^n = f(x)cx^n - g(x)cx^n$ for every positive integer n.*

PROOF. Suppose $f(x) = \sum a_i x^i$, $g(x) = \sum b_i x^i$.

Then

$$f(x)-g(x)\cdot cx^n = \sum a_i b_i' \, x^i \cdot cx^n = \sum ca_i b_i' \, x^{n+i}$$

and

$$f(x)\,cn^n - g(x)\,cx^n = \sum ca_i x^{n+i} - \sum cb_i x^{n+i} = \sum ca_i (cb_i)' x^{n+i} =$$

$$= \sum ca_i(c'+b_i')x^{n+i} = \sum ca_i b_i' \, x^{n+i}.$$

Therefore, $f(x)-g(x)\cdot cx^n = f(x)cx^n - g(x)cx^n$ completing the proof.

Theorem 9. *The ideal I generated by cx^n is a complemented ideal of $R[x]$ for every c in R and positive integer n.*

PROOF. Suppose $f(x)cx^n, g(x)cx^n\in I$. Then $f(x)cx^n - g(x)cx^n = f(x)-g(x)\cdot cx^n\in I$ completing the proof.

Definition 10: An ideal I in a semiring R will be called a k-ideal if the following condition is satisfied: If $a\in I$, $b\in R$ and $a+b\in I$, then $b\in I$.

Lemma 11. *If $f(x), g(x)\in R[x]$, then $f(x)-g(x)+f(x) = f(x)$.*

PROOF. Suppose $f(x) = \sum a_i x^i$ and $g(x) = \sum b_i x^i$.

Then

$$f(x)-g(x)+f(x) = \sum a_i b_i' \, x^i + \sum a_i x^i = \sum (a_i b_i' + a_i) x^i =$$

$$= \sum a_i (b_i'+1) x^i = \sum a_i x^i = f(x).$$

Theorem 12. *Every k-ideal in $R[x]$ is a complemented ideal.*

PROOF: Let I be a k-ideal in $R[x]$ and $f(x), g(x)\in I$. Then $f(x)-g(x)+f(x) = f(x)$ by Lemma 11. Since I is a k-ideal and $f(x)$ is in I, $f(x)-g(x)$ is in I.

The following lemma is the critical step in our development of the basis theorem.

Lemma 13. *If $I\subset R[x]$, define $I^*\subset R^*[x]$ by*

$$I^* = \{f^*(x) = a_n x^n \oplus \cdots \oplus a_0 = \textstyle\sum \oplus a_i x^i \mid f(x) = a_n x^n + \cdots + a_0 = \sum + a_i x^i \in I\}.$$

Then (1) *I^* is an ideal in $R^*[x]$ if and only if I is a completed ideal in $R[x]$.*

(2) *$I_1\subset I_2$ implies $I_1^*\subset I_2^*$.*

(3) *$I_1^* = I_2^*$ implies $I_1 = I_2$.*

PROOF: Implications (2) and (3) are obvious. Suppose I is a complemented ideal in $R[x]$. Let $f^*(x) = \sum \oplus a_i x^i$, $g^*(x) = \sum \oplus b_i x^i \in I^*$. Then $f(x) = \sum + a_i x^i$, $g(x) = \sum + b_i x^i \in I$. Then I is a complemented ideal implies

$$f(x) - g(x) + g(x) - f(x) = \sum + a_i b_i' x^i + \sum + b_i a_i' x^i = \sum + (a_i b_i' + b_i a_i') x^i \in I.$$

Therefore

$$f^*(x) \oplus g^*(x) = \sum \oplus (a_i \oplus b_i) x^i = \sum \oplus (a_i b_i' + b_i a_i') x^i \in I^*$$

and I^* is closed

under addition. Since $f^*(x) \oplus f^*(x) = \sum \oplus (a_i \oplus a_i) x^i = 0$ for every $f(x) \in I^*$, $(I^*, \oplus)$ is a group.

Suppose $h^*(x) = \sum \oplus c_j x^j \in R^*[x]$. Then $f(x) \in I$ and $c_j x^j \in R[x]$ for every j, imply $c_j x^j \cdot f(x) = \sum + c_j a_i x^{j+i} \in I$. Hence $c_j x^j \cdot f^*(x) = \sum \oplus c_j a_i x^{j+i} \in I$. Finally, $h^*(x) \cdot f^*(x) = \sum c_j x^j \cdot f^*(x) \in I^*$ and I^* is an ideal in $R^*[x]$.

The proof that I^* is an ideal in $R^*[x]$ implies I is a complemented ideal in $R[x]$ is similar and will be omitted. (This implication is not necessary for the basis theorem).

Theorem 14. *If R is a B-semiring satisfying the ascending chain condition on ideals, then $R[x]$ satisfies the ascending chain condition on complemented ideals.*

PROOF: Suppose $I_1 \subset I_2 \subset \cdots$ is an ascending chain of ideals in $R[x]$. By Lemma 13, $I_1^* \subset I_2^* \subset \cdots$ is an ascending chain of ideals in $R^*[x]$. By Corollary 3, R^* satisfies the ascending chain condition on ideals so by Hilbert's classical result $R^*[x]$ satisfies the ascending chain condition of ideals. Hence there exists a positive integer N such that $I_n^* = I_N^*$ for every $n \geqq N$. Therefore, by Lemma 13 $I_n = I_N$ for $n \geqq N$ and thus $R[x]$ satisfies the ascending chain condition on complemented ideals.

In view of theorem 12, the following corollary is immediate.

Corollary 15: If R is a B-semiring satisfying the ascending condition on idelas, then $R[x]$ satisfies the ascending chain condition on k-ideals.

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