Abelian groups in which every neat subgroup is a direct summand

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1. Introduction

When investigating the structure of a given abelian group G it is often of value to know how certain subgroups sit in G. For example, KERTÉSZ [4] has shown that every subgroup of G is a direct summand if and only if every element in G has square-free order. FUCHS, KERTÉSZ and SZELE [3] have shown that

Theorem A. *An abelian group G has the property that all its pure subgroups are direct summands if and only if it has one of the following forms:*

(1). *G is a torsion group, each p-primary component of which is the direct sum of a bounded and a divisible group.*

(2). *$G = D + E$ where D is divisible and E is the direct sum of a finite number of mutually isomorphic rank one torsion-free groups.*

Neat subgroups are just those without proper essential extensions in the containing group and therefore, from a purely module-theoretic point of view, neatness is a more natural concept than purity. Thus we are led to classify those abelian groups having the property that every neat subgroup is a direct summand (Theorem 1). As an application of this result, we classify the quasi-injective abelian groups (Theorem 2).

2. Preliminaries

By a *group* we shall mean an additively written abelian group. If A is a subgroup of G, then A is *neat* in G if and only if $pG \cap A = pA$ for all primes p; A is *pure* in G if and only if the equation $p^n G \cap A = p^n A$ holds for all primes p and all positive integers n. A group G has *Property N* if and only if every neat subgroup of G is a direct summand. In our discussions we will often use the fact that direct summands are pure.

For any group G, tG and G_p will denote the maximal torsion subgroup of G and the p-primary component of G, respectively. We use the symbol $+$ to denote direct sums. Occasionally we will refer to

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Lemma 1. *If G satisfies Property N and if H is a direct summand of G, then H also satisfies Property N.*

PROOF. Neatness of A in H implies neatness in G. Thus $G = A+B$ for some subgroup B of G and $H = A+(B\cap H)$.

3. Torsion groups with Property N

Proposition 1. If a p-group G satisfies Property N, then either G is divisible or $(*)$ $G = H+K$ where H is the direct sum of cyclic groups of order p^n and K is the direct sum of cyclic groups of order p^{n+1}.

PROOF. By Theorem A, a p-group satisfying our property must have a decomposition $G = D+B$ where D is divisible and B is bounded. To see that one of D or B is zero, we assume the existence of a quasicyclic direct summand D' of D and a cyclic summand C of B having order p^n for some positive integer n. Let $G' = D'+C$. If $d\in D'$ is of order p^{n+1} and c is a generator of C, then the subgroup A, generated by (d, c), is neat in G' but not pure in G'. Thus A is not a direct summand of G' and G' does not satisfy Property N. According to Lemma 1, G does not satisfy our property. This contradiction forces us to conclude that either D or B is zero, i.e., either G is bounded or G is divisible.

In the case G is bounded and B_1 and B_2 are cyclic summands of G having orders p^n and p^{n+m}, respectively, we assume that $m>1$. Let b_1 generate B_1 and b_2 generate B_2. Then the subgroup of $G' = B_1+B_2$ generated by (b_1, pb_2) is neat but not pure in G'. Again, this yields a contradiction and we have forced to conclude that $m\leqq 1$.

For the converse of Proposition 1 we need

Lemma 2. *If a p-group has a decomposition $(*)$, then every neat subgroup is pure.*

PROOF. Let $\{H_i : i\in I\}$ denote the cyclic summands of H having order p^n and $\{K_j : j\in J\}$ those of K having order p^{n+1}. Let A be neat in G. Note that, since p^kG is zero for all $k\geqq n+1$, $p^kG\cap A = p^kA$ for all such k. Thus we now assume that $p^mG\cap A = p^mA$ for some m, $1\leqq m<n$. If $p^{m+1}x\in A$ for some x in G, then, by assumption, there is an $a\in A$ satisfying $px-a\in G[p^m]$. If $(a)_i$ and $(x)_i$ denote the i-th coordinates of a and x, respectively, then $p(x)_i-(a)_i\in H_i[p^m] = p^{n-m}H_i\subseteqq pH_i$. Thus $(a)_i$ is contained in pH_i for each $i\in I$. Similarly, $(a)_j$ is contained in pK_j for each $j\in J$. Consequently, there is a $g\in G$ with $pg=a$. By the neatness of A, we are guaranteed an element $a'\in A$ satisfying $pg=a=pa'$. But then $p^{m+1}x=p^{m+1}a'\in p^{m+1}A$ and A is pure in G.

We now have the converse of Proposition 1 in

Proposition 2. If a p-group is divisible or has a decomposition $(*)$, then G satisfies Property N.

PROOF. Since every neat subgroup of a divisible group is divisible, we conclude that our class contains the divisible groups. So suppose that G has a decomposition $(*)$ and that A is neat in G. Then A is pure by the previous lemma and A is a direct summand since A is bounded (KULIKOV [2]).

We are now able to give the main result of this section.

Proposition 3. A torsion group G satisfies Property N if and only if each p-primary component is divisible or has a decomposition $(*)$.

PROOF. The necessity follows from Lemma 1 and Proposition 1. For the sufficiency suppose that A is neat in G and that each G_p is divisible or enjoys a decomposition $(*)$. As A_p is neat in G_p for each p, Proposition 2 implies $G_p = A_p + B_p$ for some subgroup B_p and thus $G = A + B$ where B is the direct sum of the complements B_p.

4. Arbitrary groups with Property N.

Proposition 4. If $G = D + E$ where D is divisible and E is the direct sum of a finite number of mutually isomorphic rank one torsion-free groups, then G satisfies Property N.

PROOF. Let A be neat in G. Since tA is divisible, $A = tA + B$ where B is torsion-free. As $tG \cap B$ is zero, we may argue in the usual way that $G = tG + H$ where B is a subgroup of H. Now B is neat, hence pure, in the torsion-free H. From Theorem A we conclude that $H = B + H'$ and as a consequence, $G = tG + H = (tA + G') + (B + H') = A + (G' + H')$; that is, G has Property N

Theorem 1. *An abelian group G has Property N if and only if G has one of the following forms:*

(1′). *G is a torsion group in which, for each prime p, G_p is either divisible or else of the form $H + K$ where H is a direct sum of cycic groups of order p^n and K is a direct sum of cyclic groups of order p^{n+1}.*

(2). *$G = D + E$ where D is divisible and E is the direct sum of a finite number of mutually isomorphic rank one torsion-free groups.*

PROOF. Let G satisfy Property N. If G is torsion, we apply Proposition 3 and find that G is of the form (1′). If, on the other hand, G is not torsion, then Theorem A implies that G has the form (2). (We have used the fact that if G has Property N, then pure subgroups of G are direct summands.) Proposition 3 and Proposition 4 yield the converse.

5. A characterization of quasi-injective abelian groups

We define an abelian group G to be *quasi-injective* if and only if each homomorphism of any subgroup A into G can be extended to a homomorphism of G into G. A subgroup A of G is *closed* in case A has no proper essential extensions in G. It is well known that A is neat if and only if A is closed Thus the class of abelian groups whose neat subgroups split off contains the quasi-injectives This follows from

Thorem B. (FAITH [1].) *If G is quasi-injective then every closed subgroup of G is a direct summand.*

Proposition 5. If G is quasi-injective, then either G is divisible or $(**)$ G is torsion and each G_p is the direct sum of mutually isomorphic cocyclic groups.

PROOF. It suffices to apply Theorem 1 and Lemma 1 to G after showing that none of the following types of groups are quasi-injective.

(a) $G = H+K$ where H is cyclic of order p^n and K is cyclic of order p^{n+1}.

(b) G is reduced torsion-free group having rank one.

For the case (a), we let k and h generate K and H, respectively. If H' is the subgroup of G generated by pk, then we define a homomorphism from H' to G by $f(pk)=h$. It is easy to see that no extension of f could be defined at the generator k. Thus f does not extend.

In (b) we view G as a subgroup of the additive group of rationals Q and then suppose that $1/t \in Q-G$ while $0 \neq m/n \in G$. Letting A denote the subgroup of G generated by $m^2 t/n$ we map A into G by defining $f(m^2 t/n)=m/n$. Assume that f extends to an endomorphism F of G and that $F(m)=x \in G$. Then $f(m^2 t)=m$ while $F(m^2 t)=mtx$. We now conclude that $x=1/t \in G$, contrary to assumption. Consequently, G is not quasi-injective.

Proposition 6. If G satisfies $(**)$ then G is quasi-injective.

PROOF. Let A be a subgroup of G and f a homomorphism from A into G. Then, for each prime p, we have induced homomorphisms $f_p: A_p \to G_p$. As G is the direct sum of its components G_p, extending each f_p to an endomorphism F_p of G_p is sufficient to extend f to an endomorphism of G. Now since divisible groups are quasi-injective, we need only consider the components of G which are direct sums of cyclic groups of order, say, p^n. Let D denote the minimal injective extension of G_p. Then D is a divisible p-group whose fully invariant subgroups are those of the form $D[p^k]$ for k a positive integer. Consequently G_p is fully invariant in D. The injectivity of D allows us to extend f_p to an endomorphism F of D. Setting F_p equal to the restriction of F to G_p yields the extension we need. Thus G is quasi-injective.

Proposition 5 and Proposition 6 give us

Thorem 2. *G is quasi-injective if and only if G is divisible or G satisfies $(**)$.*

As a final note we observe that those groups in which every subgroup is neat are precisely the elementary groups. This follows from the argument given in Theorem 4 of [3].

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