Operator semigroups with applications to semirings

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1. Introduction

Y. AKIZUKI [1] proved the well-known theorem that a commutative ring with an identity satisfies the descending chain condition if and only if it satisfies the ascending chain condition and every proper prime ideal is maximal. The purpose of this paper is to obtain analogues of the Second and Third Isomorphism Theorems, Schreier's Refinement Theorem, and the Jordan—Hölder Theorem for operator semigroups, and use these results to extend Akizuki's Theorem to a large class of semirings.

2. Operator Semigroups and Quotient Structures

Many different definitions of an operator semigroup appear in the literature. The class of operator semigroups considered in this paper is contained in the class studied by A. W. Goldie [6], and is defined as follows:

Definition 1. 1. An *operator semigroup* is a triple $(G, M, .)$ consisting of an associative semigroup with identity 1, an operator set M, and a mapping from $G\times M$ into G such that

(1) $(g_1 g_2)m=(g_1 m)(g_2 m)$,

(2) $1m=1$

for each $g_i\in G$, $m\in M$. This system will be referred to as an *M-semigroup G*.

Definition 1. 2. Let G be an M-semigroup. A subset H of G is called an *M-subsemigroup* of G if

(1) $h_1 h_2\in H$,

(2) $1\in H$, where 1 is the identity of G,

(3) $h_1 m\in H$,

for each $h_i\in H$, $m\in M$.

Definition 1. 3. Let G_1 and G_2 be M-semigroups. A mapping $\eta: G_1\to G_2$ is called a *homomorphism* provided

(1) $(g_1 g_2)\eta=(g_1\eta)(g_2\eta)$,

(2) $(g_1 m)\eta=(g_1\eta)m$,

for each $g_i\in G_1$, $m\in M$. If η is one-to-one and onto G_2, η is called an *isomorphism,* and G_1 and G_2 are said to be *isomorphic*, denoted $G_1\cong G_2$.

Definition 1.4. Let G_1 and G_2 be M-semigroups and $\eta: G_1 \to G_2$ a homomorphism. Then $\eta^{-1}(\{1\}) = \{g \in G_1 \mid g\eta = 1\}$ is called the *kernel* of η, and denoted $\ker \eta$.

Of special interest to P. DUBREIL [5] and N. CHAPTAL [4] was a subsemigroup H of a semigroup G such that $hg \in H$, $h \in H$ implies $g \in H$. S. BOURNE [3], M. HENRIKSON [7], D. R. LATORE [9], and P. J. ALLEN [2] studied ideals in a semiring where the additive structure of the ideal satisfied the above condition, and it is their terminology that is used here.

Definition 1.5. Let H be an M-subsemigroup of an M-semigroup G. H is called a *$k-M$-subsemigroup* of G (or is k in G) provided:

If $g \in G$, $h \in H$, then gh or $hg \in H$ implies that $g \in H$.

If N and H are subsets of an M-semigroup G, then $NH = \{nh \mid n \in N, h \in H\}$. The cosets $\{g\}H$ and $H\{g\}$ are denoted by gH and Hg, respectively.

Definition 1.6. An M-subsemigroup H of an M-semigroup G is said to be *normal* in G, denoted $H \lhd G$, if $gH = Hg$ for each $g \in G$.

The proof of the following well-known lemma is straightforward and is omitted.

Lemma 1.7. *Let G_1 and G_2 be M-semigroups and $\eta: G_1 \to G_2$ a homomorphism onto G_2. Then*

(1) $\ker \eta$ *is a $k-M$-subsemigroup of G_1.*

(2) *If H_1 is an M-subsemigroup of G_1, then $H_1\eta$ is an M-subsemigroup of G_2. Furthermore, $H_1 \lhd G_1$ implies $H_1\eta \lhd G_2$.*

(3) *If H_2 is an M-subsemigroup of G_2, then $\eta^{-1}(H_2)$ is an M-subsemigroup of G_1. Furthermore, H_2 is k in G_2 if and only if $\eta^{-1}(H_2)$ is k in G_1.*

In the theory of operator groups it is well-known that the kernel of a homomorphism is normal and that an operator group is normal in itself. The following example shows that these facts are not in general true for operator semigroups.

Example 1.8. Let $G = \{1, a, b\}$ and let multiplication in G be defined by the following table:

	1	a	b
1	1	a	b
a	a	a	a
b	b	b	b.

Taking $M = \emptyset$, G is an M-semigroup. Since $aG = \{a\}$ and $Ga = \{a, b\}$, G is not normal in G. The mapping $\eta: G \to \{1\}$ defined by $g\eta = 1$, for each $g \in G$, is a homomorphism with kernel G. Hence $\ker \eta$ is not normal in G.

Throughout this paper, quotient structures are studied with respect to the following congruence, which was first defined by Dubreil for commutative semigroups. Let H be a normal M-subsemigroup of an M-semigroup G. Define a relation R on G as follows: $g_1 R g_2$ if there exist $h_i \in H$ such that $g_1 h_1 = g_2 h_2$. Since $H \lhd G$, it follows that R is a congruence. If $g \in G$, then $[g]$ denotes the congruence class determined by g, i.e., $[g] = \{x \in G \mid xRg\}$. Then $G/R = \{[g] \mid g \in G\}$, together with the binary composition $[g_1][g_2] = [g_1 g_2]$, is a semigroup. Since this congruence is dependent on H, it is standard to denote G/R by G/H. Let $g_i \in G$ such that $[g_1] = [g_2]$. There exist $h_i \in H$ such that $g_1 h_1 = g_2 h_2$. Hence $(g_1 h_1)m = (g_2 h_2)m$, where $m \in M$, and it follows that $(g_1 m)(h_1 m) = (g_2 m)(h_2 m)$. Therefore $[g_1 m] = [g_2 m]$. This shows that the operator multiplication defined by $[g]m = [gm]$ is well-defined. The identity of G/H is $[1]$, and $[1]m = [1m] = [1]$. $([g_1][g_2])m = ([g_1 g_2]m) = [(g_1 g_2)m] = [(g_1 m)(g_2 m)] = [g_1 m][g_2 m]$. Consequently, with this operator multiplication G/H is an M-semigroup, called an M-quotient semigroup. In particular, if G and H are groups, the above development yields the standard M-factor group $G/H = \{gH \mid g \in G.\}$ In the general case Bourne and LaTore did not characterize the congruence classes, but did observe that they need not be cosets. The following lemma presents a characterization of the congruence classes.

Lemma 1.9. *Let H be a normal M-subsemigroup of an M-semigroup G. Let $[g] \in G/H$ and $g_i \in [g]$. Then*

(1) $g_1 H \cap g_2 H \neq \emptyset$,

(2) $g_1 \in [g]$ *implies* $g_1 H \subseteqq [g]$,

(3) $[g] = \bigcup_{g_i \in [g]} g_i H.$

Proof. (1) Since g_1 and g_2 are in the same congruence class, there exist $h_i \in H$ such that $g_1 h_1 = g_2 h_2$.

(2) Let $g_1 h \in g_1 H$. Since $(g_1 h)1 = g_1 h$, $g_1 h$ and g_1 are in the same congruence class, i.e., $g_1 h \in [g]$.

(3) From (2) it follows that $\bigcup_{g_i \in [g]} g_i H \subseteqq [g]$.

Since $g_i \in g_i H$, equality follows.

Consequently, any congruence class is characterized as the union of a collection of cosets $\mathscr{B} = \{g\, H\}$, where

$$g_k H,\ g_j H \in \mathscr{B},\ g_n H \notin \mathscr{B} \quad \text{imply} \quad g_k H \cap g_j H \neq \emptyset,\ g_n H \cap g_k H = \emptyset.$$

As can be seen by Example 1.8, if H is not normal in G, then $g_1 H \cap g_3 H \neq \emptyset$ and $g_2 H \cap g_3 H \neq \emptyset$ do not imply that $g_1 H \cap g_2 H \neq \emptyset$.

Definition 1.10. A homomorphism η from an M-semigroup G_1 onto an M-semigroup G_2 is called *maximal* if for each $a \in G_2$ there exists $g_a \in \eta^{-1}(\{a\})$ such that $g \in \eta^{-1}(\{a\})$ implies $g \ker \eta \subseteqq g_a \ker \eta$.

Lemma 1.11. *If η is a maximal homomorphism from an M-semigroup G_1 onto an M-semigroup G_2 such that* $\ker \eta$ *is normal in G_1, then each $[g] \in G_1/\ker \eta$ is a coset.*

Proof. If $g_i \in [g]$, then $g_i \eta = g\eta = a \in G_2$. There exists $g_a \in \eta^{-1}(\{a\})$ such that $x \ker \eta \subseteqq g_a \ker \eta$ for each $x \in \eta^{-1}(\{a\})$. Since $g_i \in \eta^{-1}(\{a\})$, $g_i \ker \eta \subseteqq g_a \ker \eta$. In particular, $g \ker \eta \subseteqq g_a \ker \eta$, and this implies that $g_a \in [g]$. From Lemma 1.9 it follows that $[g] = \bigcup_{g_i \in [g]} g_i \ker \eta \subseteqq g_a \ker \eta \subseteqq [g]$. Therefore $[g] = g_a \ker \eta$.

A less restrictive class of homomorphisms is now defined.

Definition 1.12. Let η be a homomorphism from an M-semigroup G_1 onto an M-semigroup G_2. η is called *semimaximal* if $g_1 \eta = g_2 \eta$ implies $g_1 \ker \eta \cap g_2 \ker \eta \neq \emptyset$.

In view of Lemma 1. 9, it is observed that a maximal homomorphism with normal kernel is semimaximal. The following example shows that the converse is not true in general.

Example 1. 13. Let $A=\{x|x$ is real and $1\leqq x\leqq 2\}$ and $B=\{x|x$ is real and $2<x\leqq 3\}$. Let $G = A\cup B$, and define multiplication on G as follows:

For $a_i\in A$, $b_i\in B$, $a_1a_2=\max\{a_1, a_2\}$, $b_1b_2=\max\{b_1, b_2\}$, and $a_1b_1=b_1a_1=\max\{a_1+1, b_1\}$. Then G is an M-semigroup with $M=\emptyset$. $H=\{1, 2\}$ is an M-semigroup. Define $\eta: G\to H$ by $a\,\eta=1$ for each $a\in A$, $b\eta=2$ for each $b\in B$. η is a homomorphism and $\ker\eta=A$. $a\ker\eta=\{x\in A|x\geqq a\}$ and $b\ker\eta=\{x\in B|x\geqq b\}$. Hence η is semimaximal. Since $G/\ker\eta=\{A, B\}$ and B is not a coset, η is not maximal.

Theorem 1. 14. *Let G be an M-semigroup and $H\lhd G$. Define $v: G\to G/H$ by $gv=[g]$. Then*

(1) *v is a homomorphism onto G/H,*

(2) $H\subseteqq\ker v$,

(3) *H is k in G if and only if* $H=\ker v$,

(4) *v is semimaximal,*

(5) *if H is k in G and $H\subset G$, then $G/H\neq 1$.*

Proof. The straightforward proofs of (1) and (2) are omitted.

(3) If $H=\ker v$, then H is k by Lemma 1. 7. Conversely, assume H is k in G. By (2) it suffices to show that $\ker v\subseteqq H$. Let $u\in\ker v$. Then $u\in[1]$, so there exist $h_i\in H$ such that $uh_1=h_2$. Since H is k in G_1 it follows that $u\in H$.

(4) Let $g_i\in G$, and assume $g_1v=g_2v$. Then $[g_1]=[g_2]$. By Lemma 1. 9, $g_1H\cap g_2H\neq\emptyset$. Thus v is semimaximal, since $H\subseteqq\ker v$.

(5) Since $\ker v=H\subset G$, it follows that $G/H\neq 1$.

Definition 1. 15. The map $v: G\to G/H$ given in Theorem 1. 14 is called the *natural homomorphism* of G onto G/H.

Theorem 1. 16. (Fundamental Theorem of Homomorphisms.) *Let η be a semimaximal homomorphism from an M-semigroup G_1 onto an M-semigroup G_2 such that* $\ker\eta\lhd G_1$. *Then* $G_1/\ker\eta\cong G_2$.

Proof. $\bar\eta: G_1/\ker\eta\to G_2$ defined by $[g]\bar\eta=g\eta$ is a homomorphism onto G_2. If $[g_1]\bar\eta=[g_2]\bar\eta$, then $g_1\eta=g_2\eta$, and semimaximality implies that $g_1\ker\eta\cap g_2\ker\eta\neq\emptyset$. Hence $[g_1]=[g_2]$, and $\bar\eta$ is one-to-one.

Corollary 1. 17. Let H be a normal M-subsemigroup of an M-semigroup G, and let $v: G\to G/H$ be the natural homomorphism. If $\ker v\lhd G$, then $G/\ker v\cong G/H$.

Proof. Applying the Fundamental Theorem to the semimaximal homomorphism v gives the desired result.

The following lemmas are needed to prove an analogue of the Lattice Theorem.

Lemma 1. 18. *Let η be a semimaximal homomorphism from an M-semigroup G_1 onto an M-semigroup G_2, and let $H\lhd G$. If H is k in G and* $\ker\eta\subseteqq H$, *then*

(1) $H=\eta^{-1}(H\eta)$,

(2) $H\eta$ is k in G_2.

PROOF. (1) Clearly $H \subseteq \eta^{-1}(H\eta)$. Let $g \in \eta^{-1}(H\eta)$. Then $g\eta \in H\eta$, so there exists $h \in H$ such that $h\eta = g\eta$. Since η is semimaximal, $h \ker \eta \cap g \ker \eta \neq \emptyset$, and since $\ker \eta \subseteq H$, $hH \cap gH \neq \emptyset$. Consequently, $g \in H$, since H is k in G.

(2) Let $h\eta \in H\eta$ and $g\eta \in G_2$, where $h\eta g\eta \in H\eta$. Thus $h, hg \in \eta^{-1}(H\eta) = H$. Since H is k in G_1, it follows that $g \in H$ and hence that $g\eta \in H\eta$. Similarly $g\eta h\eta \in H\eta$ implies $g\eta \in H\eta$.

Lemma 1. 19. *Let η be a semimaximal homomorphism from an M-semigroup G_1 onto an M-semigroup G_2, and let $\{H\}$ be the collection of all $k-M$-subsemigroups of G_1 that contain* ker η. *The mapping $H \to H\eta$ is one-to-one of $\{H\}$ onto the set of all $k-M$-subsemigroups of G_2.*

PROOF. That $H \to H\eta$ is well-defined follows from Lemma 1. 18, and Lemma 1. 7 implies the mapping is onto. Let $H_1, H_2 \in \{H\}$, where $H_1\eta = H_2\eta$. Lemma 1. 18 yields $H_1 = H_2$. Therefore the mapping is one-to-one.

Theorem 1. 20. (Lattice Theorem.) *Let G be an M-semigroup and H a normal $k-M$-subsemigroup of G. Then any $k-M$-subsemigroup of G/H is of the form N/H, where N is a $k-M$-subsemigroup of G containing H. If N_1 and N_2 are distinct $k-M$-subsemigroups of G containing H_1 then N_1/H and N_2/H are distinct $k-M$-subsemigroups of G/H. If $N \lhd G$, then $N/H \lhd G/H$.*

PROOF. Apply Lemma 1. 19 to the natural homomorphism $v: G \to G/H$. Any $k-M$-subsemigroup of G/H is of the form Nv, where N is a $k-M$-subsemigroup of G and contains H. Let $g \in [n]$, where $n \in N$. Then there exists $h_i \in H$ such that $gh_1 = nh_2$. Since $H \subseteq N$ and N is k in G, it follows that $g \in N$. Then $Nv = \{[n] | n \in N\} = \{[n] \cap N | n \in N\} = N/H$. The last remark follows from Lemma 1. 7.

Lemma 1. 21. *Let η_1 be a semimaximal homomorphism from an M-semigroup G_1 onto an M-semigroup G_2 and η_2 a semimaximal homomorphism from G_2 onto an M-semigroup G_3. Then $\eta_1\eta_2$ is a semimaximal homomorphism.*

PROOF. Let $g_1, g_1' \in G$, and assume $g_1\eta_1\eta_2 = g_1'\eta_1\eta_2$. Then $(g_1\eta_1)\eta_2 = (g_1'\eta_1)\eta_2$, so that $(g_1\eta_1) \ker \eta_2 \cap (g_1'\eta_1) \ker \eta_2 \neq \emptyset$. Hence there exist $u_1\eta_1, u_1'\eta_1 \in \ker \eta_2$ such that $(g_1\eta_1)(u_1\eta_1) = (g_1'\eta_1)(u_1'\eta_1)$, which implies $(g_1u_1)\eta_1 = (g_1'u_1')\eta_2$. Therefore $(g_1u_1) \ker \eta_1 \cap (g_1'u_1') \ker \eta_1 \neq \emptyset$. Since $\ker \eta_1 \subseteq \ker \eta_1\eta_2$, it follows that

$$(g_1u_1) \ker \eta_1\eta_2 \cap (g_1'u_1') \ker \eta_1\eta_2 \neq \emptyset.$$

n_1, $u_1' \in \ker \eta_1\eta_2$, which implies $g_1 \ker \eta_1\eta_2 \cap g_1' \ker \eta_1\eta_2 \neq \emptyset$.

Theorem 1. 22. (First Isomorphism Theorem.) *Let η be a semimaximal homomorphism from an M-semigroup G_1 onto an M-semigroup G_2. Let H be a normal $k-M$-subsemigroup of G_1 that contains* ker η. *Then $G_1/H \cong G_2/H\eta$.*

PROOF. Lemma 1. 7 and Lemma 1. 18 imply $H\eta \lhd G_2$. Let v be the natural homomorphism from G_2 onto $G_2/H\eta$. By Theorem 1. 14, $\ker v = H\eta$. Clearly $H \subseteq \ker \eta v$. Let $g \in \ker \eta v$. Then $g\eta \in \ker v = H\eta$. By Lemma 1. 18, $g \in H$. Therefore $H = \ker \eta v$. Applying the Fundamental Theorem to the semimaximal homomorphism ηv gives the desired result.

Corollary 1. 23. If N and H are normal $k-M$-subsemigroups of an M-semigroup G_1 and $N \subseteq H$, then

$$\frac{G}{H} \cong \frac{\frac{G}{N}}{\frac{H}{N}}.$$

PROOF. Applying the First Isomorphism Theorem to the natural homomorphism

$$v : G \to \frac{G}{N}$$

gives

$$\frac{G}{H} \cong \frac{\frac{G}{N}}{Hv}, \quad \text{and} \quad Hv = \frac{H}{N}$$

by the Lattice Theorem.

3. Isomorphism Theorems

Definition 2. 1. Let G_1 and G_2 be M-subsemigroups of an M-semigroup G. G_1 is said to be related to G_2 provided:

If $g_i, g_i' \in G_i$ such that $g_1 g_2 = g_1' g_2'$, then there exist $a, b \in G_1 \cap G_2$ such that $g_1 a = g_1' b$.

G_1 is said to be closely related to G_2 if G_1 is related to every M-subsemigroup of G_2. If G_1 is related (closely related) to G_2, and G_2 is related (closely related) to G_1, then G_1 and G_2 are said to be related (closely related).

Any M-semigroup is closely related to each of its M-subsemigroups, and all M-subgroups of an M- semigroup are related.

Example 2. 2. Let $G = \{1, a, b, c, d, e, f, g\}$ and let multiplication in G be given by the following table:

	1	a	b	c	d	e	f	g
1	1	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a	a
b	b	a	b	c	d	e	f	g
c	c	a	c	c	a	a	a	a
d	d	a	d	a	d	e	a	a
e	e	a	e	a	e	e	a	a
f	f	a	f	a	a	a	f	g
g	g	a	g	a	a	a	g	g.

Then G is an M-semigroup with $M = \emptyset$. Let $G_1 = \{1, b, c\}$, $G_2 = \{1, b, d, e\}$, $G_3 = \{1, b, f, g\}$. Then each G_i is a normal $k-M$-subsemigroup of G. It can be shown that G_1 is related to G_2, but not closely related, and G_2 is not related to G_1. G_2 is not related to G_3 and G_3 is not related to G_2.

Example 2. 3. Let $G=\{x|x$ is real and $0\leqq x\leqq 1\}$, and let $M=G$. Define multiplication in G by $g_1g_2=\max\{g_1, g_2\}$, and define operator multiplication by $gm=$ $=\min\{g, m\}$. Then G is an M-semigroup, and if H is an M-subsemigroup of G, then $H=\{x\in G|x\leqq r\}$ or $H=\{x\in G|x<r\}$, where $r\in G$. Then every M-subsemigroup of G is k in G and normal in G, and any pair are closely related.

The following lemmas concerning relatedness are crucial to the development in this paper.

Lemma 2. 4. *Let G_i be M-subsemigroups of an M-semigroup G such that G_1 is related to G_2. Let H be an M-subsemigroup of G_1 such that $G_1\cap G_2\subseteqq H$. Then H is related to G_2.*

Proof. $H\cap G_2 = G_1\cap G_2$, and the result follows.

Lemma 2. 5. *Let η be a semimaximal homomorphism from an M-semigroup G_1 onto an M-semigroup G_2. If H_1 and H_2 are M-subsemigroups of G_2 such that $\eta^{-1}(H_1)$ is related to $\eta^{-1}(H_2)$, then H_1 is related to H_2.*

Proof. Let $h_i, h_i'\in H_i$, where $h_1h_2=h_1'h_2'$. There exist $g_i, g_i'\in\eta^{-1}(H_i)$ such that $g_i\eta=h_i$ and $g_i'\eta=h_i'$. Then $(g_1g_2)\eta=(g_1'g_2')\eta$. Hence there exist $u_i\in\ker\eta$ such that $(g_1g_2)u_1=(g_1'g_2')u_2$ and therefore $g_1(g_2u_1)=g_1'(g_2'u_2)$. g_2u_1 and $g_2'u_2$ are both in $\eta^{-1}(H_2)$, so there exist $a, b\in\eta^{-1}(H_1)\cap\eta^{-1}(H_2)$ such that $g_1a=g_1'b$. Then $h_1(a\eta)=$ $=h_1'(b\eta)$, where $a\eta, b\eta\in H_1\cap H_2$. Therefore H_1 is related to H_2.

If G_1 and G_2 are M-subsemigroups of an M-semigroup G, and $G_2\lhd G_1G_2$, then $G_1G_2=G_2G_1$ is the M-subsemigroup generated by G_1 and G_2.

Theorem 2. 6. (Second Isomorphism Theorem.) *Let G_1 and G_2 be M-subsemigroups of an M-semigroup G such that G_1 is related to G_2, $G_2\lhd G_1G_2$, and G_1, G_2 are k in G_1G_2. Then*

(1) $(G_1\cap G_2)\lhd G_1$,

(2) $G_1G_2/G_2\cong G_1/G_1\cap G_2$.

Proof. (1) Let $g_1\in G_1$, $g_{12}\in G_1\cap G_2$. Then $g_1g_{12}=g_2g_1$, where $g_2\in G_2$, since $G_2\lhd G_1G_2$. Also $g_2\in G_1$, since G_1 is k in G_1G_2. Therefore $g_1(G_1\cap G_2)\subseteqq$ $\subseteqq(G_1\cap G_2)g_1$. The reverse inclusion is similarly proved, and it follows that

$$(G_1\cap G_2)\lhd G_1.$$

(2) Define $\eta: G_1\to G_1G_2/G_2$ by $g_1\eta=[g_1]$. η is clearly a homomorphism. An element of G_1G_2/G_2 is the union of cosets of the form $g_1g_2G_2$, where $g_i\in G_i$. Since $g_1g_2G_2\subseteqq g_1G_2$, η is onto. Clearly $G_1\cap G_2\subseteqq\ker\eta$. If $u\in\ker\eta$, then $uG_2\cap G_2\neq\emptyset$. Hence G_2 being k in G_1G_2 implies $u\in G_2$. Thus $\ker\eta = G_1\cap G_2$. If $g_1\eta=g_1'\eta$, then $g_1G_2\cap g_1'G_2\neq\emptyset$. Then G_1 related to G_2 implies $g_1(G_1\cap G_2)\cap g_1'(G_1\cap G_2)\neq\emptyset$. Hence η is semimaximal. The Fundamental Theorem now gives the conclusion.

Definition 2. 7. Let H be an M-subsemigroup of an M-semigroup G, and let $\{H\alpha\}$ be the collection of all $k-M$-subsemigroups of G that contain H. The *k-closure* of H in G, denotes $\overline{H}$, is defined by $\overline{H}=\cap\{H\alpha\}$.

Observe that $\overline{H}$ always exists, since G is k in itself.

Theorem 2. 8. *Let H be a normal M-subsemigroup of an M-semigroup G and $v: G \to G/H$ the natural homomorphism. Then*

(1) $\bar{H} = \{g \in G \mid gH \cap H \neq \emptyset\}$,

(2) $\bar{H} = \ker v$.

PROOF. (1) Let $F = \{g \in G \mid gH \cap H \neq \emptyset\}$. Clearly $1 \in F$. Let $f_i \in F$. There exist $h_i \in H$ such that $f_1 h_1 = h_3$, $f_2 h_2 = h_4$. Then $h_3 h_4 = (f_1 h_1)(f_2 h_2) = f_1 (h_1 f_2) h_2 = f_1 (f_2 h_5) h_2 = (f_1 f_2)(h_5 h_2)$. Thus $(f_1 f_2) H \cap H \neq \emptyset$, which implies $f_1 f_2 \in F$. If $m \in M$, then $(f_1 h_1) m = h_3 m$ and $(f_1 m)(h_1 m) = h_3 m$. Thus $(f_1 m) H \cap H \neq \emptyset$, so $f_1 m \in F$. Therefore F is an M-subsemigroup of G. Let $g \in G$ such that $g f_1 \in F$. Then $h_6 (g f_1) = h_7$ for some $h_6, h_7 \in H$. Now $h_6 (g f_1) h_1 = h_7 h_1$, which implies $(h_6 g)(f_1 h_1) = h_8$. Hence $(g h_9) h_3 = h_8$, and it follows that $gH \cap H \neq \emptyset$. Thus $g \in F$, and a similar argument shows that $f_1 g \in F$ implies $g \in F$. Thus F is k in G.

$\bar{H} = \cap \{H\alpha\}$, where $\{H\alpha\}$ is the collection of all $k-M$-subsemigroups of G that contain H. $F \in \{H\alpha\}$ so $\bar{H} \subseteq F$. If $f_1 \in F$, then $f_1 h_1 = h_3$, $h_i \in H$. $H \subseteq H\alpha$ for each α implies $f_1 \in H\alpha$ for each α, since each is k in G. Thus $f_1 \in \cap \{H\alpha\}$, and it follows that $F = \bar{H}$.

(2) $\ker v$ is k and contains H, so $\bar{H} \subseteq \ker v$. If $u \in \ker v$, then $uH \cap H \neq \emptyset$. Thus $u \in \bar{H}$ by (1). Therefore $\bar{H} = \ker v$.

Definition 2. 9. An M-subsemigroup H of an M-semigroup G is said to be *k-normal in G*, denoted $H \overset{k}{\lhd} G$, if both H and $\bar{H}$ are normal in G.

The next example shows that $H \lhd G$ does not imply $H \overset{k}{\lhd} G$.

Example 2. 10. Let B be the M-semigroup of Example 1. 8, and let A be a non-trivial, commutative M-semigroup ($M = \emptyset$). Let 0 denote an element not in $A \cup B$, and let $G_1 = A \cup \{0\}$, $G_2 = B \cup \{0\}$. Define $00 = 0$ and $x0 = 0x = 0$ for each $x \in A \cup B$. Then G_1 and G_2 are M-semigroups. Let $G = \{(g_1, g_2) \mid g_i \in G_i\}$, and define $(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2')$. G is then an M-semigroup. $H = \{(g_1, 0) \mid g_1 \in G_1\} \cup \{(1, 1)\}$ is an M-subsemigroup of G, and since $(g_1, 0)(x, y) = (g_1 x, 0) = (x g_1, 0) = (x, y)(g_1, 0)$, $H \lhd G$. Clearly $\bar{H} = G$. $(0, a)G = \{(0, 0), (0, a)\}$ and

$$G(0, a) = \{(0, 0),\ (0, a),\ (0, b)\},$$

so that $G = \bar{H}$ is not normal in G.

The following corollary of Theorem 2. 8 is crucial in the proof of the Third Isomorphism Theorem.

Corollary 2. 11. Let H be an M-subsemigroup of an M-semigroup G. If $H \overset{k}{\lhd} G$, then $G/H \cong G/\bar{H}$.

PROOF. Corollary 1. 17 and Theorem 2. 8 give the conclusion.

Theorem 2. 12. (Third Isomorphism Theorem.) *Let G be an M-semigroup, and let G_1, G_2 be $k-M$-subsemigroups of G that are closely related. Let $G_1' \lhd G_1$, $G_2' \lhd G_2$, and further assume that $(G_1 \cap G_2')G_1' \overset{k}{\lhd} (G_1 \cap G_2)G_1'$, $(G_1' \cap G_2)G_2' \overset{k}{\lhd} (G_1 \cap G_2)G_2'$. Then*

$$\frac{(G_1 \cap G_2)G_1'}{(G_1 \cap G_2')G_1'} \cong \frac{(G_1 \cap G_2)G_2'}{(G_1' \cap G_2)G_2'}.$$

PROOF. Let F be the k-closure of $(G_1 \cap G_2')G_1'$ in $(G_1 \cap G_2)G_1'$. Since F is an M-subsemigroup of G_1, G_2 is related to F. $F \cap G_2 \subseteq G_1 \cap G_2 \subseteq G_2$, so $G_1 \cap G_2$ is related to F by Lemma 2. 4. From $(G_1 \cap G_2')G_1' \subseteq F \subseteq (G_1 \cap G_2)G_1'$, it follows that $(G_1 \cap G_2)F = (G_1 \cap G_2)G_1'$. The Second Isomorphism Theorem applied to $G_1 \cap G_2$ and F yields $(G_1 \cap G_2) \cap F = G_2 \cap F \lhd G_1 \cap G_2$ and

$$\frac{G_1 \cap G_2}{G_2 \cap F} \cong \frac{(G_1 \cap G_2)F}{F} = \frac{(G_1 \cap G_2)G_1'}{F}.$$

Corollary 2. 11 gives

$$\frac{(G_1 \cap G_2)G_1'}{F} \cong \frac{(G_1 \cap G_2)G_1'}{(G_1 \cap G_2')G_1'}.$$

Thus

$$\frac{(G_1 \cap G_2)G_1'}{(G_1 \cap G_2')G_1'} \cong \frac{G_1 \cap G_2}{G_2 \cap F}. \tag{1}$$

The following argument will show $(G_1 \cap G_2')G_1' \cap G_2 \lhd G_1 \cap G_2$. Let $g \in G_1 \cap G_2$, $h \in (G_1 \cap G_2')G_1' \cap G_2$. Then $gh = h'g$, where $h' \in (G_1 \cap G_2')G_1'$, since $(G_1 \cap G_2')G_1' \lhd G_1 \cap G_2$. Since $g, h \in G_2$ and G_2 is k, it is also true that $h' \in G_2$. Thus

$$g[(G_1 \cap G_2')G_1' \cap G_2] \subseteq [(G_1 \cap G_2')G_1' \cap G_2]g,$$

and the reverse inclusion is proved similarly.

$F \cap G_2$ is the k-closure of $(G_1 \cap G_2')G_1' \cap G_2$ in $G_1 \cap G_2$, for $F \cap G_2$ is k in $G_1 \cap G_2$ and contains $(G_1 \cap G_2')G_1' \cap G_2$. Thus $F \cap G_2$ contains the k-closure. Conversely, let $x \in F \cap G_2$. Since F is the k-closure of $(G_1 \cap G_2')G_1'$, there exist $g, g' \in (G_1 \cap G_2')G_1'$ such that $xg = g'$. G_2 is related to $(G_1 \cap G_2')G_1'$, so there exist $a, b \in (G_1 \cap G_2')G_1' \cap G_2$ such that $xa = b$. Hence x is in the k-closure of $(G_1 \cap G_2')G_1' \cap G_2$.

Therefore, since $G_2 \cap F \lhd G_1 \cap G_2$,

$$\frac{G_1 \cap G_2}{G_2 \cap F} \cong \frac{G_1 \cap G_2}{(G_1 \cap G_2')G_1' \cap G_2}. \tag{2}$$

It will now be shown that $(G_1 \cap G_2')G_1' \cap G_2 = (G_1 \cap G_2')(G_1' \cap G_2)$. Let $ab \in (G_1 \cap G_2')G_1' \cap G_2$, where $a \in G_1 \cap G_2'$, $b \in G_1'$. Then $ab \in G_2$, so $b \in G_2$. Thus $b \in G_1' \cap G_2$, which implies $(G_1 \cap G_2')G_1' \cap G_2 \subseteq (G_1 \cap G_2')(G_1' \cap G_2)$. The reverse inclusion is clear.

Combining (1) and (2) yields

$$\frac{(G_1 \cap G_2)G_1'}{(G_1 \cap G_2')G_1'} \cong \frac{G_1 \cap G_2}{(G_1 \cap G_2')(G_1' \cap G_2)}. \tag{3}$$

A symmetrical argument proves

$$\frac{(G_1 \cap G_2)G_2'}{(G_1' \cap G_2)G_2'} \cong \frac{(G_1 \cap G_2)}{(G_1' \cap G_2)(G_1 \cap G_2')}. \tag{4}$$

It is now seen that $(G_1 \cap G_2')(G_1' \cap G_2) = (G_1' \cap G_2)(G_1 \cap G_2')$. Let

$$ab \in (G_1 \cap G_2')(G_1' \cap G_2),$$

where $a \in G_1 \cap G_2'$, $b \in G_1' \cap G_2$. Then $ab = b'a$, where $b' \in G_1'$, because $G_1' \lhd G_1$. Since $ab \in G_2$ $a \in G_2$, it follows that $b' \in G_2$. Thus $(G_1 \cap G_2')(G_1' \cap G_2) \subseteqq (G_1' \cap G_2)(G_1 \cap G_2')$. A similar argument shows the reverse inclusion.

Combining (3) and (4) now gives the desired result.

Definition 2. 13. Let $G = G_1 \supseteqq G_2 \supseteqq \cdots \supseteqq G_s \supseteqq G_{s+1} = 1$ be a sequence of M-subsemigroups of an M-semigroup G such that $G_{i+1} \lhd G_i$. The sequence is called a *normal series* for G. The M-quotient semigroups G_i/G_{i+1} are called the *factors* of the series, and the G_i are the *terms*. If each G_i is k in G, the series is called a *normal k-series*. Two normal series of G are said to be *equivalent* if there exists a one-to-one correspondence between the factors of the two series such that the corresponding factors are isomorphic. A normal series is a *refinement* of a second normal series if all of the terms of the latter are included in those of the former.

Theorem 2. 14. (Schreier's Refinement Theorem.) *Let*

(1) $G = G_1 \supseteqq G_2 \supseteqq \cdots \supseteqq G_{s+1} = 1,$

(2) $G = H_1 \supseteqq H_2 \supseteqq \cdots \supseteqq H_{t+1} = 1$

be two normal k-series of an M-semigroup G such that each G_i *and* H_j *are closely related. Further assume that*

$$(G_i \cap H_{j+1})G_{i+1} \overset{k}{\lhd} (G_i \cap H_j)G_{i+1} \quad \text{and} \quad (G_{i+1} \cap H_j)H_{j+1} \overset{k}{\lhd} (G_i \cap H_j)H_{j+1}.$$

Then the two series have equivalent refinements.

Proof. Let

$$G_{ij} = (G_i \cap H_j)G_{i+1},$$
$$H_{ji} = (G_i \cap H_j)H_{j+1}.$$

Then

$$\begin{aligned}(1') \quad G &= G_{11} \supseteqq G_{12} \supseteqq \cdots \supseteqq G_{1,t+1}\\ &= G_{21} \supseteqq G_{22} \supseteqq \cdots \supseteqq G_{2,t+1}\\ &\cdots\\ &= G_{s1} \supseteqq G_{s2} \supseteqq \cdots \supseteqq G_{s,t+1} = 1,\end{aligned}$$

$$\begin{aligned}(2') \quad G &= H_{11} \supseteqq H_{12} \supseteqq \cdots \supseteqq H_{1,s+1}\\ &= H_{21} \supseteqq H_{22} \supseteqq \cdots \supseteqq H_{2,s+1}\\ &\cdots\\ &= H_{t1} \supseteqq H_{t2} \supseteqq \cdots \supseteqq H_{t,s+1} = 1\end{aligned}$$

are refinements of (1) and (2), respectively. By the Third Isomorphism Theorem applied G_i, H_j, G_{i+1}, H_{j+1}, it follows that

$$\frac{G_{ij}}{G_{i,j+1}} = \frac{(G_i \cap H_j)G_{i+1}}{(G_i \cap H_{j+1})G_{i+1}} \cong \frac{(G_i \cap H_j)H_{j+1}}{(G_{i+1} \cap H_j)H_{j+1}} = \frac{H_{ji}}{H_{j,i+1}}.$$

Therefore (1′) and (2′) are equivalent.

Definition 2. 15. A *composition series* for an M-semigroup G is a normal series

$$G=G_1\supset G_2\supset\cdots\supset G_{s+1}=1$$

with the property that each G_i is maximal in G_{i+1}; i.e., if $H\lhd G_i$ and $G_{i+1}\subseteqq H\subseteqq G_i$, either $H=G_i$ or $H=G_{i+1}$. If each term is k in G, the series is called a *composition k-series.*

In view of Theorem 1. 14, a composition k-series has no trivial factors.

Theorem 2. 16. (Jordan—Hölder.) *Let*

$$G=G_1\supset G_2\supset\cdots\supset G_{s+1}=1, \tag{1}$$

$$G=H_1\supset H_2\supset\cdots\supset H_{t+1}=1, \tag{2}$$

be two composition k-series for an M-semigroup G such that each G_i and H_j are closely related. Further assume that $(G_i\cap H_{j+1})G_{i+1}\overset{k}{\lhd}(G_i\cap H_j)G_{i+1}$ *and*

$$(G_{i+1}\cap H_j)H_{j+1}\overset{k}{\lhd}(G_i\cap H_j)H_{j+1}.$$

Then the two series are equivalent.

Proof. No factor in either series $=1$. A refinement can be obtained only by inserting duplicates. Therefore a reminement has the same factors $\neq 1$ as the series which it refines. By Schreier's Refinement Theorem (1) and (2) have equivalent refinements. In the one-to-one correspondence between the factors of the refinements with paired factors isomorphic, the factors of (1) and (2) must be paired. Hence $t=s$ and the series are equivalent.

In particular, let G be a commutative M-semigroup such that any two M-subsemigroups of G are related. Then any two normal k-series for G have equivalent refinements, and any two composition k-series for G are equivalent.

Definition 2. 17. An M-semigroup G satisfies the *descending chain condition* (DCC) provided:

$$G_1\supseteqq G_2\supseteqq\cdots\supseteqq G_n\supseteqq G_{n+1}\supseteqq\cdots$$

is a descending sequence of M-subsemigroups of G implies there exists N such that $n\geqq N$ implies $G_n=G_N$.

G satisfies the *ascending chain condition* (ACC) provided:

$$G_1\subseteqq G_2\subseteqq\cdots\subseteqq G_n\subseteqq G_{n+1}\subseteqq\cdots$$

is a sequence of M-subsemigroups of G implies there exists N such that $n\geqq N$ implies $G_n=G_N$.

Theorem 2. 18. *Let G be a commutative M-semigroup with the property that any two M-subsemigroups of G are related and each is k in G. Then G has a composition series if and only if G satisfies the* ACC *and the* DCC.

Proof. Assume that G satisfies the ACC and the DCC. Let $H\neq 1$ be any term of a normal series for G. Either 1 is maximal in H or there exists an M-subsemigroup H_1 of H such that $1\subset H_1\subset H$. In the latter case either H_1 is maximal in H or there exists an M-subsemigroup H_2 of H such that $H_1 \subset H_2 \subset H$. This process must stop after a finite number of steps, for otherwise the ACC would be contradicted. Thus any term $\neq 1$ of a normal series must contain a maximal M-subsemigroup. Then G contains a maximal M-subsemigroup G_2, G_2 a maximal M-subsemigroup G_3, etc. By the DCC, $G_{s+1}=1$ for s finite. Hence

$$(1) \qquad G = G_1 \supset G_2 \supset \cdots \supset G_{s+1} = 1$$

is a composition series.

Conversely, assume that G has the composition series (1). If either the ACC or DCC fail, then there exists a sequence

$$(2) \qquad G = H_1 \supset H_2 \supset \cdots \supset H_{s+1} \supset H_{s+2} = 1$$

of M-subsemigroups of G. Then (1) and (2) have equivalent refinements. This implies that there are more than s non-trivial factors of (1), a contradiction.

4. Semirings and semimodules

Definition 3. 1. A set R together with two associative binary compositions called addition and multiplication (denoted by $+$ and $.$, respectively) is a *semiring* provided:

(1) addition is commutative,

(2) there exists $0 \in R$ such that $x+0 = x$ and $x0=0x=0$ for each $x \in R$,

(3) multiplication distributes over addition both from the left and from the right.

Definition 3. 2. If R is a commutative semiring and $R-\{0\}$ is a multiplicative group, then R is called a *semifield*.

Definition 3. 3. A subset I of a semiring R is called an *ideal* if $a, b \in I$ and $r \in R$ imply $a+b$, ar, $ra \in I$.

Definition 3. 4. An ideal I in a semiring R is called *prime* if $a, b \in R$ and $ab \in I$ imply either $a \in I$ or $b \in I$.

If $I_1, I_2, \ldots, I_n$ are ideals in a semiring R, then the ideal which they generate is denoted by $I_1 I_2 \ldots I_n$ and called the product of the ideals.

The proofs of the following three lemmas are the same as those given in ring theory and are omitted.

Lemma 3. 5. *Let R be a commutative semiring with identity. Then R is a semifield if and only if R has no proper, non-trivial ideals.*

Lemma 3. 6. *Let R be a commutative semiring with identity. If I_1 is an ideal of R that is not prime, then there exist ideals I_2, I_3 of R such that $I_1 \subset I_2$, $I_1 \subset I_3$, and $I_2 I_3 \subseteqq I_1$.*

Lemma 3. 7. *Let R be a commutative semiring with identity that satisfies the* ACC. *Then every ideal of R contains a product of prime ideals.*

Definition 3. 8. An R-semimodule is a commutative R-semigroup M (written additively) such that the operator set R is a semiring and in addition to the operator conditions

(1) $(x+y)a = xa+ya$,

(2) $0a=0$,

it is also true that

(3) $x(a+b) = xa+xb$,

(4) $x(ab)=(xa)b$,

(5) $x0=0$,

for each $a, b \in R$, $x, y \in M$. If R has an identity 1 and if $x1=x$, for each $x \in M$, then M is called *unitary*.

Note that the R-semigroup R, where R is a semiring, is an R-semimodule, and is unitary if R has an identity.

Lemma 3. 9. (Modular Law.) *Let H, L, and N be R-subsemimodules of an R-semimodule M such that H is k in M and $L \subseteqq H$. Then $H \cap (L+N) = L+(H \cap N)$.*

PROOF. Let $x \in H \cap (L+N)$. Then $x = y+z$, where $y \in L$, $z \in N$, and $y+z \in H$. Now $y \in H$ and H is k in M, so $z \in H$. Thus $z \in H \cap N$, and it follows that $H \cap (L+N) \subseteqq \subseteqq L+(H \cap N)$. The reverse inclusion is clear.

Theorem 3. 10. *Let M be an R-semimodule such that any R-submodule of M is k. Let N be an R-subsemimodule of M. Then M satisfies the* ACC (DCC) *if and only if the* ACC (DCC) *holds in both N and M/N.*

PROOF. If the ACC holds in M, it clearly must hold in N, and by the Lattice Theorem, also in M/N.

Assume that N and M/N satisfy the ACC. If L, L' are L'-R-subsemimodules of M such that $L \subseteqq L'$, $L+N=L'+N$, $L \cap N = L' \cap N$, then $L' = L' \cap (L'+N) = L' \cap \cap (L+N) = L+(L' \cap N) = L+(L \cap N) = L$. Let $\{L_i\}$ be an ascending sequence of R-subsemimodules of M. To show that the sequence becomes constant after a finite number of terms, it suffices to show that the ascending sequences $\{L_i+N\}$, $\{L_i \cap N\}$ become constant after a finite number of terms. For the latter sequence this follows from the ACC in N. For the former it follows from the ACC in M/N, in view of the Lattice Theorem.

The theorem is similarly proved for the DCC.

Corollary 3. 11. Let M be an R-semimodule such that any R-subsemimodule of M is k in M. Let $M_1, M_2, \ldots, M_n$ be R-subsemimodules of M such that $M = M_1 + +M_2+\cdots+M_n$ and M_n is related to $M_1+M_2+\cdots+M_{n-1}$. If each M_i satisfies the ACC (DCC), so does M.

PROOF. The proof is by induction. The result is clear for $n=1$. Let $n=2$; i.e., $M = M_1+M_2$. By Theorem 3. 10, it suffices to show that M/M_1 satisfies the ACC (DCC).

$$\frac{M}{M_1} = \frac{M_1+M_2}{M_1} \cong \frac{M_2}{M_1 \cap M_2}$$

and the last R-semimodule satisfies the ACC (DCC) by Theorem 3. 10. Thus the ACC (DCC) holds in M/M_1.

Assume the result true for $n-1$. Consider $M = M_1+M_2+\cdots+M_n$. The above argument for $n=2$ with M_1 replaced by $(M_1+M_2+\cdots+M_{n-1})$ and M_2 by M_n gives the desired result.

Lemma 3. 12. *Let M be a unitary R-semimodule, where R is a semifield. Assume that any two R-semimodules of M are related and that each is k in M. If M satisfies the* ACC, *then M satisfies the* DCC.

PROOF. Let $x_1 \in M$, where $x_1 \neq 0$. $x_1 R$ is an R-subsemimodule of M. If $M \neq x_1 R$, there exists $x_2 \in M - x_1 R$. Then $x_1 R + x_2 R$ is an R-subsemimodule of M properly containing $x_1 R$. By this process an increasing sequence

$$x_1 R \subset x_1 R + x_2 R \subset x_1 R + x_2 R + x_3 R \subset \cdots$$

of R-subsemimodules of M is obtained. By the ACC, the sequence must become constant. Hence $M = x_1 R + x_2 R + \cdots + x_r R$. Each $x_i R$ has no proper, non-trivial R-subsemimodules, and so satisfies the DCC. Thus M satisfies the DCC by Corollary 3. 11.

Definition 3. 13. The annihilator of an R-semimodule M, denoted $A(M)$, is defined by $A(M) = \{r \in R | mr = 0, \text{ for each } m \in M\}$.

It is easily seen that $A(M)$ is an ideal in R.

Lemma 3. 14. *Let M be an R-semimodule and* $I \subseteq A(M)$. *Define operator multiplication of M by R/I to be* $m[r] = mr$. *Then M is an R/I-semimodule.*

PROOF. Let $r_1, r_2 \in [r]$. There exist $i_1, i_2 \in I$ such that $r_1 + i_1 = r_2 + i_2$, and hence $mr_1 = m(r_1+i_1) = m(r_2+i_2) = mr_2$, for $m \in M$. Thus the operator multiplication is well-defined. It is straightforward to verify the operator conditions.

C. HOPKINS [8] proved that in a ring with identity, the DCC implies the ACC, which is part of the necessity of Akizuki's Theorem. An analogue of the sufficiency of Akizuki's Theorem concludes this paper, and the following example shows that under the same conditions Hopkins' result need not be valid.

Example 3. 15. Let R be the set of non-negative integers $\cup \{\infty\}$, and define $a+b = \max\{a, b\}$, $ab = \min\{a, b\}$. Then R is a commutative semiring with an identity. A proper ideal I of R is of the form $I = \{a \in R | a < r\}$, where $r \in R$. Every ideal is k in R and any two ideals are related. Moreover, R satisfies the DCC, but not the ACC.

Theorem 3. 16. (AKIZUKI.) *Let R be a commutative semiring with identity, with the property that any two ideals of R are related and each is k in R. If R satisfies the* ACC *and every prime ideal of R not equal to R is maximal, then R satisfies the* DCC.

PROOF. Since R satisfies the ACC, by Lemma 3. 7 every ideal contains a product of prime ideals. Hence $\{0\}$ is a product of prime, and hence maximal, ideals $I_1, I_2, \ldots, I_n$. Then $\{0\} = I_1 I_2 \ldots I_n$. Consider the sequence $R \supseteq I_1 \supseteq I_1 I_2 \supseteq I_1 I_2 I_3 \supseteq \ldots \supseteq I_1 I_2 \ldots I_n = \{0\}$. The R-semimodule $I_1 \ldots I_{i-1}/I_1 \ldots I_{i-1} I_i$ is annihilated by I_i, and so by Lemma 3. 14 it may be considered to be an R/I_i-semimodule. R/I_i is a semifield by Lemma 3. 5. Since R satisfies the ACC, so does $I_1 \ldots I_{i-1}/I_1 \ldots I_{i-1} I_i$. Observe that the R-subsemimodules and the R/I_i-subsemimodules of $I_1 \ldots I_{i-1}/I_1 \ldots I_{i-1} I_i$ are the same. They are also R-subsemimodules of $R/I_1 \ldots I_{i-1} I_i$, and so any two are related and each is k in $I_1 \dots I_{i-1}/I_1 \dots I_{i-1} I_i$. Thus $I_1 \dots I_{i-1}/I_1 \dots I_{i-1} I_i$ satisfies the DCC by Lemma 3. 12. Thus by Theorem 2. 18, $I_1 \dots I_{i-1}/I_1 \dots I_{i-1} I_i$ has a composition series. It must be of the form

$$\frac{I_1 \dots I_{i-1}}{I_1 \dots I_{i-1} I_i} \supset \frac{J_1}{I_1 \dots I_{i-1} I_i} \supset \frac{J_2}{I_1 \dots I_{i-1} I_i} \supset \dots \supset \frac{I_1 \dots I_{i-1} I_i}{I_1 \dots I_{i-1} I_i} = 0,$$

where $I_1 \dots I_{i-1} \supset J_1 \supset J_2 \supset \dots \supset I_1 \dots I_{i-1} I_i$, and each ideal is maximal in the preceding ideal. Hence R has a composition series. Therefore, by Theorem 2. 18, R satisfies the DCC.

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