Near-rings without nilpotent elements

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The primary purpose of this paper is to classify near-rings which have no non-zero nilpotent elements and which satisfy the minimum condition on R-subgroups. Rings with these two properties are just direct sums of division rings. The near-ring situation is more complex. Even the structure of finite near integral domains is not completely known.

It is shown in this paper that a near-ring R with D.C.C. on R-subgroups and which contains no (non-zero) nilpotents is a direct sum of a finite number of R_i such that each R_i has no divisors of zero, no proper R_i-subgroups, has at least one idempotent, and every idempotent is a left identity. Furthermore R is von Neumann regular and if R has a non-zero right distributive element, then each R_i is a near-field and the additive group of R is abelian. If R is finite, then there exist integer $n>1$ such that $x^n=x$ for all $x\in R$; n being independent of x. Consequently a Boolean near-ring with D.C.C. on R-subgroups is the direct sum of near-rings each with the property $xy=y$ for each y if $x\neq 0$.

To illustrate the un-ringlike behavior of near-rings without nilpotents, examples are given of a finite near-ring without zero divisors which is not a near-field and satisfies $x^4=x$ for each x and of a near-ring without nilpotents which has an identity but which is not regular (and hence not the direct product or sum of near-fields); the latter example, of course, does not satisfy the minimum condition on R-subgroups.

1. *Preliminary remarks*

In this paper *near-ring* will mean left near-ring with a two-sided zero, i.e., $0r=r0=0$. A normal subgroup (subgroup) S is a left ideal (N-subgroup) of a near-ring N if $NS\subseteq S$. (Here $AB=\{ab: a\in A, b\in B\}$ for any subsets A and B of N.) A *right ideal* is a normal subgroup S such that $(n_1+s)n_2-n_1n_2\in S$ for each $s\in S$, $n_1, n_2\in N$. An *ideal* is a subset which is both a left and a right ideal. Ideals are exactly the kernels of near-ring homomorphisms. A simple near-ring is one in which the only ideals are (0) and N. In any near-ring N the right annihilating set of an element x, $\mathrm{Ann}(x)=\{n\in N: xn=0\}$, is a right ideal. If x is an idempotent, then $\mathrm{Ann}(x)+xN=N$ and $\mathrm{Ann}(x)\cap xN=(0)$.

We will use the expressions "N has no nilpotent elements" and "N has no divisors of zero" to mean N has no non-zero objects of these types. An N-subgroup

G is said to be nilpotent if there exists a positive integer n such that every product of n elements from G is zero, i.e., $G^n=(0)$.

A near-ring N is said to be (von Neumann) *regular* if for each $x\in N$, there exists $x'\in N$ such that $xx'x=x$. (The structure of regular near-rings is investigated in [5].) A near-ring is Boolean if every element is idempotent. Note that in a regular near-ring xx' and $x'x$ are non-zero idempotents if $x\neq 0$.

An element d in a near-ring N is a *distributive* element if $(a+b)d = ad+bd$ for each $a, b\in N$. If N contains a multiplicative semigroup S which generates N additively and such that every element of S is distributive, then N is called a *distributively generated* (d.g.) near-ring. For details on d.g. near-rings the reader is referred to the primal paper by FRÖHLICH [4].

In the sequel we will make use of the fact that the additive group of a near-field is commutative [9].

2. *General structure theory*

Let R be a near-ring without nilpotent elements. Then if $ab=0$ it follows that $ba=0$ and hence $arb=0$ for each $r\in R$. Thus for each non-zero $x\in R$, the annihilator, Ann (x), is an ideal of R. Since $x^2\neq 0$ we have Ann $(x)\neq R$.

If R is simple, then Ann $(x)=(0)$ and hence R has no divisors of zero. It immediately follows that every non-zero idempotent of R is a left identity.

Consider R now to be simple, have no nilpotents, and satisfy D.C.C. on R-subgroups. In this case for any non-zero x we have $xR\supseteq x^2R\supseteq\dots$ is a descending chain of R-subgroups; hence there exists n such that $x^nR=x^{n+1}R$ and since R has no divisors of zero it follows that $R=xR$. Thus $(R-\{0\},\ \cdot)$ is a right simple semigroup (CLIFFORD and PRESTON [3, p. 6]) and R has only (0) and R as R-subgroups. BLACKETT [1] has shown that a near-ring N with D.C.C. on N-subgroups and with no non-zero nilpotent N-subgroups is the (group) direct sum of minimal right ideals of the form e_iN, where e_i is an idempotent. Applying this to R we have there exists a non-zero idempotent $e\in R$; recall that e must be a left identity.

Since $xR=R$ we have for each $x\neq 0$ there exists $y\in R$ such that $xy=e$, i.e. $(R-\{0\},\ \cdot)$ is a right group. So $xyx=ex=x$ and we see that R is regular. Since R has no divisors of zero we have that y is unique if x' is right distributive.

If R has a non-zero right distributive element d, then for each $r\in R$, $rd=rdd'd$ and hence $(r-rdd')d=0$ or $r=rdd'$. Thus the idempotent dd' is a right identity and hence is the identity; so $(R-\{0\},\ \cdot)$ is a group and R must be a near-field. It follows that $(R,+)$ is commutative.

If R is d.g., then R is a d.g. near-ring with commutative addition and hence must be a ring [4]; thus in this case R will be a division ring.

We summarize the above results in the following

Theorem 2. 1. *If R is a simple near-ring without nilpotent elements and R satisfies the* D.C.C. *on R-subgroups, then*

(1) *R has no divisors of zero;*
(2) *$xR=R$ for each non-zero x;*
(3) *every non-zero idempotent of R is a left identity; R has at least one such idempotent;*

(4) $(R-\{0\}, \cdot)$ *is a right group;*
(5) *R has no proper R-subgroups;*
(6) *R is regular;*
(7) *if R has a non-zero right distributive element, then R is a near-field and hence* $(R, +)$ *is commutative;*
(8) *if R is d.g., then R is a division ring.*

Besides division rings and near-fields there are other near-rings satisfying the hypotheses of Theorem 2. 1. The following is one such, for others see Clay's tables [2].

Example 2. 2. Clay (0, 1, 2, 4, 4, 2, 1) on C_7

	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	4	1	5	2	6	3
4	0	4	1	5	2	6	3
5	0	2	4	6	1	3	5
6	0	1	2	3	4	5	6

Note that $x^4=x$ for each x and that this is not a near-field, although the addition is commutative.

We next consider the case where R is not necessarily simple. It is clear that if R has no nilpotent elements, then R has no non-zero nilpotent R-subgroups. We then can apply Blackett's decomposition theorem which is stated next for easy reference.

Lemma 2. 3. (BLACKETT [1].) *If N is a near-ring with* D.C.C. *on N-subgroups and no non-zero nilpotent N-subgroups, then N is the direct sum of a finite number of ideals, where each summand N_i is a simple near-ring with* D.C.C. *on N_i-subgroups.*

Thus if R has no nilpotent elements and has D.C.C. on R-subgroups, then R is the direct sum of near-rings as described in Theorem 2. 1. Since the direct sum of regular near-rings is is regular we see that R is regular.

The following lemma is useful; the proof is straightforward and will be omitted.

Lemma 2. 4. *If N is a near-ring with a non-zero right distributive element and $N = N_1 \oplus N_2$, as a direct sum of ideals, then N_1 and N_2 each contains a non-zero right distributive element. If N is d.g., then N_1 and N_2 are d.g. near-rings.*

From this lemma and Theorem 2. 1 we immediately obtain

Theorem 2. 5. *Let R be a near-ring without nilpotents and R have D.C.C. on R-subgroups.*

(1) *If R has a non-zero right distributive element, then R is the direct sum of near-fields and* $(R, +)$ *is commutative.*
(2) *If R is d.g., then R is the direct sum of division rings.*
(3) *R is regular and has a left identity.*

In the special case where R is Boolean the structure of the summands can be made more precise. (The following result was originally obtained by R. COURVILLE.)

Theorem 2. 6. *If R is a Boolean near-ring with* D.C.C. *on R-subgroups, then $R = R_1 \oplus \ldots \oplus R_k$, as a direct sum of ideals, where each R_i is a trivial near-ring, i.e., $ab = b$ for each b and each non-zero a.*

PROOF. We need only consider the simple summands R_i. Since each non-zero idempotent in R_i is a left identity we see that the multiplication must be trivial.

It is of interest that if one considers Boolean near-rings without a two-sided zero this result does not hold.

A consequence of Theorem 2. 6 is that a d.g. Boolean near-ring with D.C.C. is the direct sum of two element fields and hence is a Boolean ring.

We next turn to the case where R is finite and has no nilpotents. As before we consider R simple first. In this case R has no zero divisors so for each non-zero $r \in R$ there exists an integer $n > 1$ (perhaps depending on r) such that $r^n = r$. Hence r^{n-1} is a left identity. Because of the trivial multiplication we cannot say anything in general about the additive structure of R. However, if R is non-trivial, then LIGH [7] has shown that $(R, +)$ must be nilpotent.

Theorem 2. 7. *If R is a finite near-ring with no nilpotent elements, then for each non-zero $x \in R$ there exists an $n > 1$, independent of x, such that $x^n = x$.*

PROOF. We first consider R to be simple. Then $x^n = x$ for each x, where n may depend on x. If $y \in R$ such that $y^m = y$, then $x^k = x$ and $y^k = y$, where $k = nm - n - m + 2$. To see this recall that x^{n-1} and y^{m-1} are idempotents and note that

$$x^k = x^{(n-1)(m-1)+1} = (x^{n-1})^{m-1} x = x^{n-1} x = x;$$

similarly for y^k. Since R is finite we can repeatedly apply this to obtain a k which will serve for all $r \in R$. We call k the *power constant* for R.

Next consider R without the simplicity restriction. Then $R = R_1 \oplus \ldots \oplus R_j$, where the R_i are simple. We show $R_1 \oplus R_2$ has the desired property and then repeat the process to obtain it for all of R. Let $r = r_1 + r_2$, where $r_i \in R_i$. Let n and m be the power constants for R_1 and R_2 respectively. Since $R_1 \oplus R_2$ is the direct sum of ideals we have that $(r_1 + r_2)x = r_1 x + r_2 x$ for each $x \in R_1 \oplus R_2$ (see HEATHERLY [6, Lemma 4. 1]). Since R_1 and R_2 are ideals this yields $(r_1 + r_2)^i = r_1^i + r_2^i$ for each positive integer i. Let $i = nm - n - m + 2$: then $(r_1 + r_2)^i = r_1^i + r_2^i = r_1 + r_2$. as above.

We conclude this section with a result that does not involve a chain condition. If N is a distributive near-ring, then the commutator subgroup N' of $(N, +)$ is nilpotent; in fact $N' \cdot N' = 0$ [8]. So a distributive near-ring either has nilpotents or must be a ring.

The question arises as to whether a d.g. near-ring without nilpotent elements must be a ring.

3. *An example without* D.C.C.

The structure theory developed in Section 2 depended strongly on having the D.C.C. on R-subgroups. The general situation, without a finiteness condition, appears to be open. The following is a class of examples of near-rings without zero divisors which do not satisfy the D.C.C. on R-subgroups.