Basic submodules of an R-module

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KULIKOV introduced the notion of basic subgroup of an abelian p-group which has proved to be one of the most important notions in the theory of p-groups of arbitrary power.

Basic submodules can be defined in any module over the ring of p-adic integers, or, more generally, over any discrete valuation ring. FUCHS (Notes on abelian group II. *Acta Math. Acad. Sci. Hung.* **11** (1960), 117—125) has given a generalization of basic subgroups to any group so that it coincides with the old concept whenever the group is primary. In the general case of groups, to every prime p, one can define p-basic subgroups, where in the definition the prime p plays a distinguished role. In this paper, it is our aim to generalize this concept of basic subgroups in any group to basic submodule in a module over a Dedekind ring.

A *Dedekind ring* R is an integral domain in which every ideal ($\neq 0, \neq R$) is uniquely a product of prime ideals. Moreover, every prime ideal of a Dedekind ring is maximal and every ideal is finitely generated. During our investigation here, we will consider R to be a Dedekind ring and we shall exclusively consider modules over the fixed Dedekind ring R which we shall call for the sake of brevity R-modules.

Definition 1: The nonzero elements $a_1, a_2, \ldots, a_n$ of an R-module M will be called *linearly independent* or simply *independent*, if

(1) $$\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_n a_n = 0 \qquad (\lambda_i \in R)$$

implies

(2) $$\lambda_1 a_1 = 0, \ldots, \lambda_n a_n = 0$$

If $O(a)$ denotes the order of $a \in M$, i.e. the ideal of R consisting of all $\lambda \in R$ with $\lambda a = 0$, then the conclusion (2) can also be written in the form

$$\lambda_1 \in O(a_1), \ldots, \lambda_n \in O(a_n).$$

If every element is of order 0, then naturally,

$$\lambda_1 = 0, \ldots, \lambda_n = 0.$$

Remark 1: An infinite set of elements is independent if each of its finite subsets is independent, therefore, the independence of the set $\{a_\lambda\}_{\lambda \in \Lambda}$ is equivalent to the condition that the submodule $\langle \ldots, a_\lambda, \ldots \rangle$ generated by all the a_λ is the direct sum of the cyclic submodules Ra_λ:

$$\langle \ldots, a_\lambda, \ldots \rangle = \cdots \oplus Ra_\lambda \oplus \cdots$$

2. If each of the a_λ ($\lambda \in \Lambda$) is of order 0, then the independence of the set $\{a_\lambda\}_{\lambda \in \Lambda}$ is equivalent to the condition that the submodule $\langle \dots, a_\lambda, \dots \rangle$ is a free R-module with a_λ as free generators.

3. Independence is a property of finite character. Therefore, in every R-module M there exist maximal independent sets. Moreover, every independent set of M can be extended to a maximal independent set of M.

Definition 2: Let M be an R-module and P a fixed prime ideal of R. We shall call a subset $\{a_\lambda\}_{\lambda \in \Lambda}$ of M not containing zero, *P-independent*, if for any finite subset $a_1, a_2, \dots, a_k$, a relation

$$n_1 a_1 + \dots + n_k a_k \in P^r M \, (n_i a_i \neq 0, \; n_i \in R)$$

implies $\langle n_i \rangle \subseteqq P^r$, where r is any positive integer, $i = 1, 2, \dots, k$.

Lemma 1. *A P-independent set is independent.*

Proof. Suppose $\{a_\lambda\}_{\lambda \in \Lambda}$ is P-independent and $n_1 a_1 + \dots + n_k a_k = 0$, where $n_i a_i \neq 0$ ($n_i \in R$). Then $n_1 a_1 + \dots + n_k a_k \in P^r M$ for every positive integer r, whence each $\langle n_i \rangle \subseteqq P^r \forall i$, for every power of P. Hence $\langle n_i \rangle \subseteqq \bigcap_r P^r = 0.$[1]) Thus $\langle n_i \rangle = 0$ and hence $n_i = 0$. Therefore, the arising contradiction establishes the statement.

Definition 3: A submodule N of an R-module M is called *P-pure*, if

$$P^r N = N \cap P^r M, \quad \text{for} \quad r = 1, 2, \dots$$

where P is a fixed prime ideal of the ring R.

Definition 4: Let M be an R-module. If $0 \neq a \in M$ and $O(a) = P^k$, where P is a prime ideal, then the element a is said to be of *P-power order*.

Lemma 2. *The submodule N generated by a P-independent subset $\{a_\lambda\}_{\lambda \in \Lambda}$ of M is P-pure in M. Conversely, if an independent set containing but elements of P-power order and/or of order* 0 *generates a P-pure submodule, then it is P-independent.*

Proof. Suppose that $h \in N \cap P^r M$, where N is the submodule generated by a P-independent set $\{a_\lambda\}_{\lambda \in \Lambda}$ of M. Then $h = n_1 a_1 + \dots + n_k a_k \in P^r M$, where we may assume that $n_i a_i \neq 0$ ($n_i \in R$, $\forall i$). By P-independence there exist ideals B_i such that $\langle n_i \rangle = P^r B_i$ ($i = 1, 2, \dots, k$). Let $n_i = \sum_j \alpha_j \beta_{i_j}$ ($i = 1, 2, \dots, k$, $\alpha_j \in P^r$, $\beta_{i_j} \in B_i$). Now

$$h = \sum_{i=1}^{k} n_i a_i = \sum_i \Big(\sum_j \alpha_j \beta_{i_j} \Big) a_i = \sum_j \alpha_j \Big(\sum_i \beta_{i_j} a_i \Big) \in P^r N.$$

i.e. $P^r N = N \cap P^r M$. Hence N is P-pure in M.

Conversely, let $\{a_\lambda\}_{\lambda \in \Lambda}$ be an independent set such that a_λ are of P-power order and/or of order 0. Let $N = \langle \dots, a_\lambda, \dots \rangle$ is P-pure in M; to show that $\{a_\lambda\}_{\lambda \in \Lambda}$ is P-independent.

[1] See Zariski and Samuel: Commutative Algebra I, pp. 216, Corollary 1.

Let $h = n_1 a_1 + \ldots + n_k a_k \in P^r M$ $(n_i a_i \neq 0,\ n_i \in R)$. By P-purity of N in M, we have

$$n_1 a_1 + \ldots + n_k a_k \in P^r N, \text{ suppose}$$

$$n_1 a_1 + \ldots + n_k a_k = \sum_i p_i b_i \qquad (p_i \in P^r,\ b_i \in N)$$

where,

$$b_i = n_{i_1} a_1 + n_{i_2} a_2 + \ldots + n_{i_k} a_k \qquad (n_{i_j} \in R,\ j = 1, 2 \ldots, k).$$

Hence,

$$n_1 a_1 + \ldots + n_k a_k = \sum_i p_i (n_{i_1} a_1 + \ldots + n_{i_k} a_k) =$$

$$= \sum_i p_i n_{i_1} a_1 + \sum_i p_i n_{i_2} a_2 + \ldots + \sum_i p_i n_{i_k} a_k .$$

By independence of $\{a_\lambda\}_{\lambda \in \Lambda}$, we have

$$n_j a_j = \sum_i p_i n_{i_j} a_j .$$

If $O(a_j) = 0$, then $n_j - \sum_i p_i n_{i_j} = 0$, hence

$$\langle n_j \rangle \subseteqq P^r, \text{ i.e. } \{a_\lambda\}_{\lambda \in \Lambda} \text{ is } P\text{-independent.}$$

If $O(a_j)$ is P-power, then

$$n_j - \sum_i p_i n_{i_j} \in P^r, \text{ hence}$$

$$\langle n_j \rangle \subseteqq P^r.$$

This proves the Lemma.

Definition 5: We call an R-module M *P-divisible* if it satisfies $PM = M$, where P is a fixed prime ideal of the ring R.

Definition 6: A submodule B of an R-module M will be called a P-basic submodule of M, if

(i) *B is the direct sum of cyclic P-modules and/or cyclic submodules of order* 0,
(ii) *B is a P-pure submodule of M,*
(iii) *the factor module M/B is P-divisible.*

[We emphasize that P denotes the same fixed prime ideal throughout.]

Remark. M is a P-basic submodule of itself if and only if (i) holds for M; and 0 is a P-basic submodule of M if and only if M is P-divisible.

If B is a P-basic submodule of an R-module M then B may be written as $B = \bigoplus_{n=0}^{\infty} B_n$, where B_0 is free and B_n $(n \neq 0)$ is a direct sum of cyclic P-modules of order ideal P^n.

Lemma 3. *A subset $\{a_\lambda\}_{\lambda \in \Lambda}$ of an R-module M generates a P-basic submodule B of M if and only if it is a P-independent set containing but elements of P-power order and/or of order* 0 *and $M = \langle B, P^n M \rangle$ for each n.*

PROOF. Let $\{a_\lambda\}_{\lambda\in\Lambda}$ be a P-independent set in M, then by Lemma 2, it generates a P-pure submodule B of M. Since $\{a_\lambda\}_{\lambda\in\Lambda}$ is P-independent, by Lemma 1, it is independent hence and by the given condition on a_λ's. B is a direct sum of cyclic submodules, $B=\bigoplus_{n=0}^{\infty} B_k$, where we may suppose that B_0 is free and B_k is a direct sum of cyclic modules of order ideal P^k for $k=1,2,\ldots$. Now to show M/B is P-divisible. Since $M=\langle B, P^nM\rangle$ for each n, any $g\in M$, $g\neq 0$ can be written as

$$g=b+\sum_i \alpha_i g_i$$

where $b\in B$, $\alpha_i\in P^n$, $g_i\in M$. This implies that

$$g+B=\sum_i \alpha_i(g_i+B),\ \text{i.e.,}$$

M/B is P-divisible.

Conversely, suppose $\{a_\lambda\}_{\lambda\in\Lambda}$ generates a P-basic submodule B of M. Then by Lemma 2 and from conditions (i) and (ii) of definition 6, we obtain the P-independence of the set $\{a_\lambda\}_{\lambda\in\Lambda}$. Since M/B is P-divisible, for $g\in M$, $(g\neq 0)$ we have $g+B=$ $=\sum_i \alpha_i(g_i+B)$ $(\alpha_i\in P^n, g_i\in M)$. This implies $\sum_i \alpha_i g_i - g\in B$, i.e. $\sum_i \alpha_i g_i - g = b$ for some $b\in B$. Hence $g=b+\sum_i \alpha_i g_i$. Thus $g\in\langle B, P^nM\rangle$, i.e., $M=\langle B, P^nM\rangle$ for every n.

Theorem 4. *An R-module M contains P-basic submodules B for every prime ideal P of R if* $M=\langle B, P^nM\rangle$ *for each n.*

PROOF. Every R-module M contains P-independent set $\{a_\lambda\}_{\lambda\in\Lambda}$ containing but elements of P-power order and/or of zero order. Then by given condition and using Lemma 3. $\{a_\lambda\}_{\lambda\in\Lambda}$ generates a P-basic submodule.

References

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