The upper radical construction

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Let $\mathbf{W}$ be a universal class of not necessarily associative rings. For a nonempty subclass $\mathbf{M}$ of $\mathbf{W}$ we consider the class function $\mathcal{U}\mathbf{M}=\{\mathbf{R}\in\mathbf{W} \mid \text{every } \mathbf{0}\neq\mathbf{R}/\mathbf{I}\notin\mathbf{M}\}$. The class $\mathbf{M}$ is said to be *s-complete* if it satisfies the property: If $\mathbf{R}\in\mathbf{M}$, then for every $\mathbf{0}\neq\mathbf{I}\Delta\mathbf{R}$ there exists some $\mathbf{0}\neq\mathbf{I}/\mathbf{J}\in\mathbf{M}$ (we use the notation $\mathbf{I}\Delta\mathbf{R}$ to mean $\mathbf{I}$ is an ideal of $\mathbf{R}$). It is well known (see e.g. [2, pp. 6—7]) that if $\mathbf{M}$ is s-complete, $\mathcal{U}\mathbf{M}$ is a radical class called the *upper radical class* determined by $\mathbf{M}$. However, the converse fails as shown in Example 1. In Theorem 1 we give necessary and sufficient conditions on a class $\mathbf{M}$ in order that $\mathcal{U}\mathbf{M}$ be a radical class, a hereditary radical class, or a hypernilpotent radical class. In the remainder of this note we consider subclasses which determine the same upper radical class.

Recall that a nonempty subclass $\mathbf{P}$ of $\mathbf{W}$ is a radical class in $\mathbf{W}$ if it has the properties (i) $\mathbf{P}$ is homomorphically closed, and (ii) if $\mathbf{R}\in\mathbf{W}$ with $\mathbf{R}\notin\mathbf{P}$, then there exists $\mathbf{I}\Delta\mathbf{R}$ such that $\mathbf{0}\neq\mathbf{R}/\mathbf{I}\in\mathcal{S}\mathbf{P}=\{\mathbf{R}\in\mathbf{W} \mid \text{if } \mathbf{0}\neq\mathbf{I}\Delta\mathbf{R} \text{ then } I\notin\mathbf{P}\}$. $\mathbf{P}$ is said to be *hereditary* if whenever $\mathbf{R}\in\mathbf{P}$ then $\mathbf{I}\in\mathbf{P}$ for all $\mathbf{I}\Delta\mathbf{R}$. A hereditary radical class is called a *hypernilpotent* radical class if it contains all nilpotent rings.

Consider the following properties for a nonempty subclass $\mathbf{M}$ of $\mathbf{W}$:

(A) If $\mathbf{0}\neq\mathbf{R}\in\mathbf{M}$, there exists some $\mathbf{0}\neq\mathbf{R}/\mathbf{I}\in\mathcal{S}\mathcal{U}\mathbf{M}$.

(B) If $\mathbf{0}\neq\mathbf{I}\Delta\mathbf{R}\in\mathbf{W}$ is such that some $\mathbf{0}\neq\mathbf{I}/\mathbf{J}\in\mathbf{M}$, there exists $\mathbf{H}\Delta\mathbf{R}$ such that $\mathbf{0}\neq\mathbf{R}/\mathbf{H}\in\mathcal{S}\mathcal{U}\mathbf{M}$.

(C) If $\mathbf{0}\neq\mathbf{R}\in\mathbf{W}$ is such that $\mathbf{R}^2=\mathbf{0}$, then $\mathbf{R}\notin\mathbf{M}$.

Theorem 1. *The class $\mathcal{U}\mathbf{M}$ is a radical class in $\mathbf{W}$ if and only if $\mathbf{M}$ has property* (A). *$\mathcal{U}\mathbf{M}$ is a hereditary radical class if and only if $\mathbf{M}$ has properties* (A) *and* (B), *and is a hypernilpotent radical class if and only if $\mathbf{M}$ has properties* (A), (B), *and* (C).

PROOF. First note that for any class $\mathbf{M}$, $\mathcal{U}\mathbf{M}$ is always homomorphically closed. Suppose $\mathbf{M}$ satisfies property (A) and let $\mathbf{0}\neq\mathbf{R}\in\mathbf{W}$ with $\mathbf{R}\notin\mathcal{U}\mathbf{M}$. Then there is some $\mathbf{0}\neq\mathbf{R}/\mathbf{I}\in\mathbf{M}$, so by (A) there exists $\mathbf{0}\neq(\mathbf{R}/\mathbf{I})/(\mathbf{K}/\mathbf{I})\cong\mathbf{R}/\mathbf{K}\in\mathcal{S}\mathcal{U}\mathbf{M}$. Thus $\mathcal{U}\mathbf{M}$ is radical.

Conversely, suppose $\mathcal{U}\mathbf{M}$ is radical. If $\mathbf{0}\neq\mathbf{R}\in\mathbf{M}$, then since $\mathbf{M}\cap\mathcal{U}\mathbf{M}=\{\mathbf{0}\}$, $\mathbf{R}\notin\mathcal{U}\mathbf{M}$. Thus by (ii) there exists some $\mathbf{0}\neq\mathbf{R}/\mathbf{I}\in\mathcal{S}\mathcal{U}\mathbf{M}$.

To see the equivalence of hereditariness and property (B) in the case $\mathcal{U}\mathbf{M}$ is a radical class, first suppose $\mathbf{M}$ has property (B). Let $\mathbf{0}\neq\mathbf{R}\in\mathcal{U}\mathbf{M}$ and consider any $\mathbf{0}\neq\mathbf{I}\Delta\mathbf{R}$. If $\mathbf{I}\notin\mathcal{U}\mathbf{M}$, there exists $\mathbf{0}\neq\mathbf{I}/\mathbf{J}\in\mathbf{M}$, so by (B) there exists $\mathbf{0}\neq\mathbf{R}/\mathbf{H}\in\mathcal{S}\mathcal{U}\mathbf{M}$, contradicting (i). Thus $\mathcal{U}\mathbf{M}$ is hereditary.

On the other hand suppose $\mathcal{U}\mathbf{M}$ is hereditary. Let $\mathbf{0}\neq\mathbf{R}\in\mathbf{W}$ and let $\mathbf{0}\neq\mathbf{I}\Delta\mathbf{R}$ be such that some $\mathbf{0}\neq\mathbf{I}/\mathbf{J}\in\mathbf{M}$. Then $\mathbf{I}\notin\mathcal{U}\mathbf{M}$, so by the hereditariness of $\mathcal{U}\mathbf{M}$, $\mathbf{R}\notin\mathcal{U}\mathbf{M}$. Thus by (ii), $\mathbf{0}\neq\mathbf{R}/\mathbf{H}\in\mathcal{S}\mathcal{U}\mathbf{M}$ for some $\mathbf{H}\Delta\mathbf{R}$.

Clearly if $\mathscr{U}\mathbf{M}$ is a hypernilpotent radical class then $\mathbf{M}$ satisfies property (C). To see the converse, let $\mathbf{0} \neq \mathbf{R} \in \mathbf{W}$ be nilpotent and suppose $\mathbf{R} \notin \mathscr{U}\mathbf{M}$. Then there exists $\mathbf{I} \Delta \mathbf{R}$ such that $\mathbf{0} \neq \mathbf{R}/\mathbf{I} \in \mathscr{S}\mathscr{U}\mathbf{M}$. Since $\mathbf{R}$ is nilpotent, $\mathbf{R}/\mathbf{I}$ is also with index of nilpotency $m > 1$. Let $\mathbf{K}/\mathbf{I} = (\mathbf{R}/\mathbf{I})^{m-1}$. Then $\mathbf{0} \neq \mathbf{K}/\mathbf{I} \Delta \mathbf{R}/\mathbf{I}$ and $(\mathbf{K}/\mathbf{I})^2 = 0$. Furthermore, any image of $\mathbf{K}/\mathbf{I}$ must also have zero multiplication. Thus by (C), no nonzero image of $\mathbf{K}/\mathbf{I}$ lies in $\mathbf{M}$, contradicting $\mathbf{R}/\mathbf{I} \in \mathscr{S}\mathscr{U}\mathbf{M}$. Therefore $\mathbf{R} \in \mathscr{U}\mathbf{M}$, and so $\mathscr{U}\mathbf{M}$ is hypernilpotent.

Example 1: Let $\mathbf{N}$ denote the class of all nil rings. Then $\mathbf{N}$ is a radical class in the universal class $\mathbf{W}$ of all not necessarily associative rings (see e.g. [2, pp. 18—19]) with semisimple class $\mathscr{S}\mathbf{N} = \{\mathbf{R} \in \mathbf{W} \mid \mathbf{R}$ has no nil ideals$\}$. Let $\mathbf{K}$ be any non-nil ring which has a nonzero nil ideal $\mathbf{J}$ (e.g., take $\mathbf{K} = \mathbf{F}[x]/(x)^2$, $\mathbf{F}$ any field). Let $\mathbf{M} = \mathscr{S}\mathbf{N} \cup \cup \{\mathbf{K}\}$. Then $\mathbf{M}$ is not *s*-complete, for every image of $\mathbf{J}$ must also be a nil ring and hence not in $\mathbf{M}$.

Let $\mathbf{0} \neq \mathbf{R} \in \mathbf{M}$, so that either $\mathbf{R} \in \mathscr{S}\mathbf{N}$ or $\mathbf{R} \cong \mathbf{K}$.

Case 1: If $\mathbf{R} \in \mathscr{S}\mathbf{N}$, $\mathbf{R}$ has no nil ideals. Thus if $\mathbf{H}$ is any nonzero ideal of $\mathbf{R}$, $\mathbf{N}(\mathbf{H}) \neq \mathbf{H}$, where $\mathbf{N}(\mathbf{H})$ is the nil radical of $\mathbf{H}$. Hence $\mathbf{0} \neq \mathbf{H}/\mathbf{N}(\mathbf{H}) \in \mathscr{S}\mathbf{N} \subseteq \mathbf{M}$. Therefore every nonzero ideal of $\mathbf{R}$ has a nonzero image in $\mathbf{M}$; i.e., $\mathbf{R} \in \mathscr{S}\mathscr{U}\mathbf{M}$.

Case 2: If $\mathbf{R} \cong \mathbf{K}$, then $\mathbf{N}(\mathbf{R})$ is a proper ideal of $\mathbf{R}$. Thus $\mathbf{0} \neq \mathbf{R}/\mathbf{N}(\mathbf{R}) \in \mathscr{S}\mathbf{N}$, and by Case 1 we see that $\mathbf{R}/\mathbf{N}(\mathbf{R}) \in \mathscr{S}\mathscr{U}\mathbf{M}$.

Therefore by Theorem 1, $\mathscr{U}\mathbf{M}$ is a radical class.

We give one further application of Theorem 1.

Example 2: A ring $\mathbf{K}$ is called *completely indempotent* if $(a^2) = (a)$ for all $a \in \mathbf{K}$. In a recent paper [1] ANDRUNAKIEVITCH and RJABUHIN proved the following:

Theorem. *The following conditions are equivalent for an associative ring* $\mathbf{K}$*:*

(1) $\mathbf{K}$ *is completely idempotent,*

(2) $\mathbf{K}/\mathbf{I}$ *has no nilpotent elements for all ideals* $\mathbf{I}$ *of* $\mathbf{K}$,

(3) *If* $\mathbf{K}/\mathbf{I}$ *is subdirectly irreducible it has no zero divisors, and*

(4) *Every* $\mathbf{K}/\mathbf{I}$ *is a subdirect sum of subdirectly irreducible rings with no zero divisors.*

Either of the conditions (2) or (3) can be used to provide another proof that *the class of all completely idempotent rings is a radical class in the universal class W of all associative rings:*

First let $\mathbf{M} = \{\mathbf{R} \in \mathbf{W} \mid \mathbf{R}$ contains a nonzero nilpotent element$\}$ and let $\mathbf{K} \in \mathbf{M}$. Pick $\mathbf{0} \neq a \in \mathbf{K}$ to be a nilpotent. Let $\mathbf{I} \Delta \mathbf{K}$ be maximal with respect to exclusion of a. Then $\mathbf{K}/\mathbf{I}$ is subdirectly irreducible with nonzero heart, namely the principal ideal generated by the image of a in $\mathbf{K}/\mathbf{I}$. Since every ideal of $\mathbf{K}/\mathbf{I}$ contains a nonzero nilpotent, every ideal is in $\mathbf{M}$ and thus not in $\mathscr{U}\mathbf{M}$. Hence $\mathbf{K}/\mathbf{I} \in \mathscr{S}\mathscr{U}\mathbf{M}$. Therefore from Theorem 1 it follows that $\mathscr{U}\mathbf{M}$, the class of all completely idempotent rings by (2), is a radical class.

To apply (3), let $\mathbf{M}' = \{\mathbf{R} \in \mathbf{W} \mid \mathbf{R}$ is a subdirectly irreducible ring containing a zero divisor$\}$ and let $\mathbf{K} \in \mathbf{M}'$. If $\mathbf{K}$ is completely idempotent, then [1, Theorem 1, p. 1015] $bc = 0$ for $b, c \in \mathbf{K}$ implies that $(b) \cap (c) = (0)$, contradicting the subdirect irreducibility of $\mathbf{K}$. Hence for some $a \in \mathbf{K}$, $(a^2) \neq (a)$. So if $\mathbf{I} \Delta \mathbf{K}$ is chosen to be maximal with respect to $(a^2)\subseteq\mathbf{I}$ and $a\notin\mathbf{I}$, then $\mathbf{K}/\mathbf{I}$ is subdirectly irreducible with heart $(\bar{a})$, where $\bar{a}$ is the image of a in $\mathbf{K}/\mathbf{I}$. For any $\mathbf{0}\neq\mathbf{J}\Delta\mathbf{K}/\mathbf{I}$, $\bar{a}\in\mathbf{J}$ so $\mathbf{J}$ has an ideal $\mathbf{H}$ maximal with respect to $\bar{a}\notin\mathbf{H}$. Then $\mathbf{J}/\mathbf{H}$ contains a nonzero nilpotent element and thus is in $\mathbf{M}'$. Hence $\mathbf{J}\notin\mathscr{U}\mathbf{M}'$, and so $\mathbf{K}/\mathbf{I}\in\mathscr{S}\mathscr{U}\mathbf{M}'$. So again, $\mathscr{U}\mathbf{M}'$, the class of all completely idempotent rings by (3), is seen to be a radical class.

We now consider certain subclasses $\mathbf{M}$ and $\mathbf{N}$ of the arbitrary universal class $\mathbf{W}$ for which $\mathscr{U}\mathbf{M}=\mathscr{U}\mathbf{N}$. It should be noted that the results hold whether or not $\mathscr{U}\mathbf{M}$ and $\mathscr{U}\mathbf{N}$ are radical classes.

Theorem 2. *Let* $\mathbf{M}$ *and* $\mathbf{N}$ *be nonempty subclasses of* $\mathbf{W}$ *such that one of* $\mathbf{M}$ *and* $\mathbf{N}$ *is homomorphically closed. Then* $\mathscr{U}\mathbf{M}=\mathscr{U}\mathbf{N}$ *if and only if* $\mathbf{M}$ *and* $\mathbf{N}$ *satisfy property* (a): *Every nonzero ring in* $\mathbf{M}\cup\mathbf{N}$ *has a nonzero image in* $\mathbf{M}\cap\mathbf{N}$.

PROOF. Without loss of generality assume that $\mathbf{M}$ is the homomorphically closed class.

Suppose that $\mathscr{U}\mathbf{M}=\mathscr{U}\mathbf{N}$ and let $\mathbf{0}\neq\mathbf{R}\in\mathbf{M}\cup\mathbf{N}$. If $\mathbf{R}\in\mathbf{M}$, then $\mathbf{R}\notin\mathscr{U}\mathbf{M}=\mathscr{U}\mathbf{N}$ so there exists $\mathbf{0}\neq\mathbf{R}/\mathbf{I}\in\mathbf{N}$. Thus $\mathbf{R}/\mathbf{I}\in\mathbf{M}\cap\mathbf{N}$. If $\mathbf{R}\in\mathbf{N}$, then $\mathbf{R}\notin\mathscr{U}\mathbf{M}$ so there exists $\mathbf{0}\neq\mathbf{R}/\mathbf{I}\in\mathbf{M}$. Now $\mathbf{R}/\mathbf{I}\notin\mathscr{U}\mathbf{N}$, so there exists $\mathbf{K}\Delta\mathbf{R}$, $\mathbf{I}\subseteq\mathbf{K}$, such that $\mathbf{0}\neq(\mathbf{R}/\mathbf{I})/(\mathbf{K}/\mathbf{I})\cong$ $\cong\mathbf{R}/\mathbf{K}\in\mathbf{N}$. Since $\mathbf{R}/\mathbf{K}$ is a homomorphic image of $\mathbf{R}/\mathbf{I}$, $\mathbf{R}/\mathbf{K}\in\mathbf{M}\cap\mathbf{N}$.

Conversely suppose $\mathbf{R}\notin\mathscr{U}\mathbf{M}$. Then there exists $\mathbf{0}\neq\mathbf{R}/\mathbf{I}\in\mathbf{M}$. By property (a), we can find $\mathbf{K}\Delta\mathbf{R}$, $\mathbf{I}\subseteq\mathbf{K}$, such that $(\mathbf{R}/\mathbf{I})/(\mathbf{K}/\mathbf{I})\cong\mathbf{R}/\mathbf{K}\in\mathbf{M}\cap\mathbf{N}\subseteq\mathbf{N}$. Hence $\mathbf{R}\notin\mathscr{U}\mathbf{N}$ and so $\mathscr{U}\mathbf{N}\subseteq\mathscr{U}\mathbf{M}$. Similarly $\mathscr{U}\mathbf{M}\subseteq\mathscr{U}\mathbf{N}$, and we have the desired equality.

Corollary 1: Property (a) is asufficient condition that $\mathscr{U}\mathbf{M}=\mathscr{U}\mathbf{N}$ for arbitrary classes $\mathbf{M}$ and $\mathbf{N}$.

Theorem 3. *If* $\mathbf{M}$ *and* $\mathbf{N}$ *are subclasses of* $\mathbf{W}$, *one of which consists entirely of Noetherian rings, then* $\mathscr{U}\mathbf{M}=\mathscr{U}\mathbf{N}$ *if and only if* $\mathbf{M}$ *and* $\mathbf{N}$ *satisfy property* (a).

PROOF. Without loss of generality assume that $\mathbf{M}$ is the Noetherian class.

Suppose $\mathscr{U}\mathbf{M}=\mathscr{U}\mathbf{N}$ and let $\mathbf{0}\neq\mathbf{R}\in\mathbf{M}\cup\mathbf{N}$. If $\mathbf{R}\in\mathbf{M}$, then $\mathbf{R}\notin\mathscr{U}\mathbf{N}$ so there exists $\mathbf{I}_1\Delta\mathbf{R}$ such that $\mathbf{0}\neq\mathbf{R}/\mathbf{I}_1\in\mathbf{N}$. Then $\mathbf{R}/\mathbf{I}_1\notin\mathscr{U}\mathbf{M}$ so there exists $\mathbf{I}_2\Delta\mathbf{R}$, $\mathbf{I}_1\subseteq\mathbf{I}_2$, such that $\mathbf{0}\neq(\mathbf{R}/\mathbf{I}_1)/(\mathbf{I}_2/\mathbf{I}_1)\cong\mathbf{R}/\mathbf{I}_2\in\mathbf{M}$. Continuing the process we obtain an ascending chain of ideals of $\mathbf{R}$, $(0)=\mathbf{I}_0\subseteq\mathbf{I}_1\subseteq\mathbf{I}_2\subseteq\ldots$, so since $\mathbf{R}$ is Noetherian $\mathbf{I}_n=\mathbf{I}_{n+1}$ for some $n\geqq 0$. Then $\mathbf{R}/\mathbf{I}_n=\mathbf{R}/\mathbf{I}_{n+1}\in\mathbf{M}\cap\mathbf{N}$.

If $\mathbf{R}\in\mathbf{N}$, as above we get $\mathbf{0}\neq\mathbf{R}/\mathbf{J}_1\in\mathbf{M}$. Then in the same manner we arrive at an ascending chain of ideals of $\mathbf{R}/\mathbf{J}_1$, $\mathbf{J}_1/\mathbf{J}_1\subseteq\mathbf{J}_2/\mathbf{J}_1\subseteq\mathbf{J}_3/\mathbf{J}_1\subseteq\ldots$ which must stop at some $n\geqq 1$. Then $\mathbf{R}/\mathbf{J}_n\cong(\mathbf{R}/\mathbf{J}_1)/(\mathbf{J}_n/\mathbf{J}_1)=(\mathbf{R}/\mathbf{J}_1)/(\mathbf{J}_{n+1}/\mathbf{J}_1)\cong\mathbf{R}/\mathbf{J}_{n+1}\in\mathbf{M}\cap\mathbf{N}$. Therefore property (a) is established and Corollary 1 yields the converse.

Example 3: Let $\{x_i\}$ $i=1, 2, \ldots$ be a countably infinite set of indeterminates over Z, the ring of integers, and let $\mathbf{K}=Z[x_1, x_2, \ldots]$. For each $n=1, 2, \ldots$ let $\mathbf{I}_n=(x_1, 2x_2, \ldots, nx_n)$ and define $\mathbf{K}_n=\mathbf{K}/\mathbf{I}_n$. Let $\mathbf{W}$ be any universal class containing $\mathbf{K}$ and define subclasses $\mathbf{M}$ and $\mathbf{N}$ of $\mathbf{W}$ by $\mathbf{M}=\{\mathbf{0}, \mathbf{K}_2, \mathbf{K}_4, \ldots, \mathbf{K}_{2n}, \ldots\}$ and $\mathbf{N}=\{\mathbf{0}, \mathbf{K}_1, \mathbf{K}_3, \ldots, \mathbf{K}_{2n-1}, \ldots\}$. Note that for each n, $\mathbf{K}_n$ is not Noetherian, and that if $n>m$, then $\mathbf{K}_n$ is a homomorphic image of $\mathbf{K}_m$.

Now let $\mathbf{R}\in\mathscr{U}\mathbf{M}$ and suppose that $\mathbf{R}\notin\mathscr{U}\mathbf{N}$. Then there exists $\mathbf{I}\Delta\mathbf{R}$ such that $\mathbf{0}\neq\mathbf{R}/\mathbf{I}\in\mathbf{N}$; say $\mathbf{R}/\mathbf{I}\cong\mathbf{K}_{2n-1}$. But then $\mathbf{K}_{2n}$ is a homomorphic image of $\mathbf{R}/\mathbf{I}$, and thus of $\mathbf{R}$, which is in $\mathbf{M}$, a contradiction. Hence $\mathscr{U}\mathbf{M}\subseteq\mathscr{U}\mathbf{N}$. The reverse inclusion follows similarly, so $\mathscr{U}\mathbf{M}=\mathscr{U}\mathbf{N}$.

However, $\mathbf{M} \cap \mathbf{N} = \{\mathbf{0}\}$, so property (*a*) does not hold for $\mathbf{M}$ and $\mathbf{N}$. Thus we see that the hypotheses that one class be homomorphically closed, and that one class be Noetherian, cannot be dropped from Theorems 2 and 3, respectively.

As one application of Corollary 1, we have:

Theorem 4. *Let* $\mathbf{M}$ *be a nonempty subclass of* $\mathbf{W}$. *Then* $\mathcal{U}(\mathbf{W} - \mathcal{U}\mathbf{M}) = \mathcal{U}\mathbf{M}$.

PROOF. For any $\mathbf{N} \subseteqq \mathbf{W}$, $\mathcal{U}\mathbf{N} = \mathcal{U}(\mathbf{N} \cup \{\mathbf{0}\})$. Thus it suffices to show that $\mathcal{U}\mathbf{M}' = \mathcal{U}\mathbf{M}$, where $\mathbf{M}' = (\mathbf{W} - \mathcal{U}\mathbf{M}) \cup \{\mathbf{0}\}$.

Let $\mathbf{0} \neq \mathbf{R} \in \mathbf{M}'$. Then $\mathbf{R} \notin \mathcal{U}\mathbf{M}$, so there exists $\mathbf{0} \neq \mathbf{R}/\mathbf{I} \in \mathbf{M}$. Since $\mathbf{M} \subseteqq \mathbf{M}'$, the result follows from Corollary 1.

Noting that $\mathbf{M} \subseteqq \mathbf{N}$ implies $\mathcal{U}\mathbf{N} \subseteqq \mathcal{U}\mathbf{M}$, we have:

Corollary 2: If $\mathbf{M}$ is a subclass of $\mathbf{W}$, then $\mathcal{U}\mathbf{M} = \mathcal{U}\mathbf{N}$ for any subclass $\mathbf{N}$ such that $\mathbf{M} \subseteqq \mathbf{N} \subseteqq (\mathbf{W} - \mathcal{U}\mathbf{M}) \cup \{\mathbf{0}\}$.

References

[1] B. A. ANDRUNAKIEVITCH and JU. M. RJABUHIN, Completely idempotent rings and the generalized nil radical, *Dokl. Akad. Nauk. SSSR* **201** (1971), 1015—1018. *(Russian)*

[2] N. J. DIVINSKY, Rings and radicals, *Toronto, (Ont.)*, 1965.

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