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A note on n-strongly (0-minimal) minimal quasi-ideals in (semigroups with 0) rings

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Abstract. In this paper we show that a (semigroup with 0) ring contains a completely (0-) simple ideal if, and only if, it contains a n-strongly (0-) minimal quasi-ideal. It turns out that our criterion is equivalent to the solutions presented by L. M. Gluskin and O. Steinfeld to the following problem: When does a (semigroup with 0) ring contain a completely (0-) simple ideal? In the concluding section we exhibit basic results concerning (unions) sums or n-strongly (0-) minimal quasi-ideals.

0. Introduction

In this second note on n-strongly (0-minimal) minimal quasi-ideals in (semigroups with 0) rings we examine the behaviour of these type of quasi-ideals in the face of completely (0-simple) simple ideals.

Since we set to work with associative rings and semigroups simultaneously, we denote both by R. Moreover, 0 means the zero of the ring R and the absorbing element of the semigroup R. First we recall some notation and standard terms which we shall use throughout this paper.

An additive subgroup (a non-empty subset) Q of a ring (a semigroup) R is called a *quasi-ideal* of R if $QR \cap RQ \subseteq Q$ holds. A nonzero two-sided (left, right or quasi-) ideal Q of a ring (semigroup with 0) is called *minimal (0-minimal)* if Q does not properly contain any nonzero two-sided (left, right or quasi-) ideal of R. To simplify our notions we shall use *(0-) minimal* two-sided (left, right or quasi-)ideal if we deal with both cases simultaneously. In a previous paper [7] we have introduced the following notion: A (0-) minimal quasi-ideal Q of R is called *n-strongly (0-) minimal* if the two-sided ideal (Q) generated by Q is a (0-) minimal ideal of R,

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furthermore $(Q)^2 \neq 0$. We remark that this notion coincides with Clifford's notion of a regular, strongly (0-) minimal quasi-ideal (cf. [7]), Remark 2.9). We restrict our attention to the notion of a n-strongly (0-) minimal quasi-ideal since it is of great advantage; result P_2 below amply illustrates this. A (semigrup with 0) ring R is called (0-) *simple* if it has only two two-sided ideals $\{0\}$ and R, furthermore $R^2 \neq 0$ (cf. [5], p. 40 and p. 56). A 0-simple semigroup having at least one primitive idempotent element is called *completely 0-simple* (cf. [1], p. 76). If a two-sided ideal I of a semigroup R with 0 is a completely 0-simple subsemigroup of R, then I is called a *completely 0-simple ideal* of R. Following L. M. Gluskin and O. Steinfeld [2], by a *completely simple ring* we shall mean a simple ring having at least one minimal left ideal or at least one minimal right ideal. Similarly a two-sided ideal I of ring R is a *completely simple ideal* of R if I is a completely simple ring. Note that we shall use the term *completely (0-) simple ideal* if we deal with semigroups and rings at the same time. For undefined terms and notation we refer to the monograph of O. Steinfeld [5]. With these preliminaries out of the way, we get down to the essentialities of the paper.

1. Two related problems

In paper [7] we have shown that any (0-) minimal quasi-ideal of a semiprime (semigroup with 0) ring is n-strongly (0-) minimal. This note is concerned with the following question:

Problem a) When does a (semigroup with 0) ring R contain a nonzero quasi-ideal Q such that it is n-strongly (0-) minimal?

In this direction we also mention the following known question (cf. Rich's paper [3] and Gluskin-Steinfeld's paper [2]):

Problem b) When does a (semigroup with 0) ring R contain a two-sided ideal I such that it is a completely (0)-simple ideal of R?

The first purpose of this paper is to show that these problems are related as follows.

Proposition 1.1. *Let R be a (semigroup with 0) ring, then R contains a n-strongly (0-) minimal quasi-ideal if and only if R contains a completely (0-) simple ideal.*

We shall also prove two corollaries of this proposition. (See Corollaries 1.2 and 1.3)

We begin by quoting results which prove to be very useful for us. The following result is extracted from the first parts of Theorems 1.3 and 1.7 in [2].

P_1. *Let K and L be* (0-) *minimal right and left ideals of a (semigroup with* 0) *ring R, respectively, such that $KL \neq 0$. Then $I = RKLR$ is a completely* (0-) *simple ideal of R.*

To our proofs we shall also need the following combined result of paper [7, Corollaries 2.8 and 2.10]:

P_2. *Let Q be any n-strongly* (0-) *minimal quasi-ideal of a (semigroup with* 0) *ring R, then the following assertions hold:*

i) There exist idempotents e, f in the ideal (Q) such that $Q = fRe = Re \cap fR$ where Re, fR are (0-) *minimal left, right ideals of R, respectively.*

ii) For any element $x \in R$, $xQ(Qx)$ is either zero or a n-strongly (0-) *minimal quasi-ideal of R.*

iii) Every element of Q is regular.

In order to proof our first result we especially need the converse part of [2, Proposition 1.2] combined with [2, Corollary 3.2].

P_3. *If I is a completely* (0-) *simple ideal of a (semigroup with* 0) *ring R, then $I = LK$ where L and K are* (0-) *minimal left and right ideals of R, respectively. Moreover, $I^2 = I$ and $KL \neq 0$.*

Proof of Proposition 1.1. Let I be a completely (0-) simple ideal of R. By the last result P_3, one has $I = LK$ and $KL \neq 0$, where K and L have their usual meaning. Due to (Theorem 6.7b) Theorem 6.7a of [5], the (subsemigroup) subring KL is a (0-) minimal quasi-ideal of R. Since $I = I^2$ this yields $I^2 = LKI$, whence $KI \neq 0$. The (0-) minimality of K implies $KI = K$. Therefore $KIL = KL \subseteq I$. Furthermore, the ideal (KL) coincides with the ideal I since I is (0-) minimal. Evidently then $(KL)^2 = (KL)$. From these considerations one must conclude that the quasi-ideal KL is n-strongly (0-) minimal.

Conversely, assume R contains a n-strongly (0-) minimal quasi-ideal Q. Due to result P_2 *i)* the ideal (Q) contains idempotent elements e and f such that $Q = fRe$ where fR is a (0-) minimal right ideal of R and Re is a (0-) minimal left ideal of R. Obviously $fR.Re \neq 0$. Hence in view of result P_1, above, $I = RfR.ReR$ is the required completely (0-) simple ideal of R, and is such that $I = (Q)$.

The following immediate consequence from the second part of the above proof enables us to answer Problem (a) as follows.

Corollary 1.2. *Let Q be a (0-) minimal quasi-ideal Q of a (semigroup with 0) ring R. Then Q is n-strongly (0-) minimal if and only if the corresponding ideal (Q) is completely (0-) simple.*

The last proposition gives considerably more insight if we combine it with [2, Corollaries 3.1 and 3.3.] of Gluskin–Steinfeld as follows.

Corollary 1.3. *The following conditions on a (semigroup with 0) ring R are equivalent:*

i) R has at least one (0-) minimal right ideal R and at least one (0-) minimal left ideal L such that $(LK)^2 \neq 0$,

ii) Condition i) with "$(LK)^2 \neq 0$" replaced with "$LK \neq 0$ and $KL \neq 0$,"

iii) Condition i) with "$(LK)^2 \neq 0$" replaced with "$KL \neq 0$,"

iv) R has at least one completely (0-) simple ideal,

v) R has at least one n-strongly (0-) minimal quasi-ideal Q.

Proof. For the proofs of the implications $i \Rightarrow ii)$, $ii) \Rightarrow iii)$ and $iii) \Rightarrow iv)$ we refer to the corresponding proofs given in [2]. The implication $iv) \Rightarrow v)$ is valid by Proposition 1.1. We prove $v) \Rightarrow i)$. Assume $v)$ and let $a \in Q\backslash\{0\}$. Then by [7, Theorem 2.6] Ra, aR are non-nilpotent (0-) minimal left and right ideals of R, respectively. Due to result P_2 $iii)$ $a = axa$ for some x in R. Put $e = xa$. This and the (0-) minimality of Ra imply $Ra = Re$. Clearly $xaR \neq 0$, and so we can apply Proposition 6.9 of [5]. Hence xaR is a (0-) minimal right ideal of R such that $xaR = eR$. Obviously then $(Re.eR)^2 \neq 0$. This proves $i)$. Therewith we conclude the equivalence of all conditions *i)– v)*.

2. Back to basic results

We begin with the main result of this section.

Theorem 2.1. *Let L be a left (right or two-sided) ideal of a (semigroup with 0) ring R. If L contains at least one n-strongly (0-) minimal quasi-ideal of R, then the (union) sum of all n-strongly (0-) minimal quasi-ideals of R contained in L forms a left (right or two-sided) ideal of R.*

Proof. *We prove the semigroup case*. Let U_L denote the union of all n-strongly 0-minimal quasi-ideals of R contained in L. Obviously U_L is non-empty and $U_L \subseteq L$. Let $a \in U_L$ and $r \in R$, then $a \in Q_\alpha$ for some α belonging to the index set $\wedge$. From the dual of the result P_2 *ii)* $ra \in U_L$ is clear since $rQ_\alpha \subseteq U_L \subseteq L$. Hence U_L is a left ideal of R. Likewise one can prove the remaining two cases. The proofs in the ring case run analogously.

Corollary 1.3 and the preceding result imply the following.

Corollary 2.2. *If the (semigroup with* 0*) ring* R *contains at least one* n*-strongly* (0-*) minimal quasi-ideal* Q*, then the following assertions hold:*

i) The completely (0-*) simple ideal* (Q) *is equal to the (union) sum of all* n*-strongly* (0-*) minimal quasi-ideals of* R *contained in* (Q)*.*

ii) The (union) sum U' *of all* n*-strongly* (0-*) minimal quasi-ideals of* R *forms a two-sided ideal of* R*.*

Remarks 2.3. (a) Assertion *ii)* of the last result can be considered as an analogue to [6, Corollary 2.7] and [4, Corollary 2.6].

(b) Recall that any 0-minimal quasi-ideal of a semiprime semigroup (with 0) is n-strongly 0-minimal (cf. [7], Proposition 2.3). Let K be a 0-minimal two-sided ideal of a semiprime semigroup R, and let K contains at least one n-strongly 0-minimal quasi-ideal Q. Evidently, $(Q) = K$. Apply the dual of assertion *i)* of the last result on K and, so we conclude at once Theorems 7.9 and 7.12 of [5].

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