Additive relations

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Introduction

To give the concept of additive relations, we begin with some facts from analysis and algebra (see [5] and [6]).

A subset S of the Cartesian product of two sets is called a relation. The domain and the range of S are $\{x: \exists y:(x, y) \in S\}$ and $\{y: \exists x:(x, y) \in S\}$, respectively. The image of a set A under S is $S(A)=\{y: \exists x \in A:(x, y) \in S\}$. S is said to be a function if $(x, y_1), (x, y_2) \in S$ implies $y_1 = y_2$.

A subset of an algebraic structure X is called a complex of X. For every $x \in X$ $\{x\}$ is identified with x. If A and B are complexes of X and $*$ is an operation in X, then $A * B = \{a * b : a \in A, b \in B\}$.

In § 1. we define: A relation S with domain an additive groupoid X and range contained in an additive groupoid Y is additive if for all $a, b \in X$

$$S(a+b) = S(a)+S(b).$$

In § 2. we deal with the representation of additive relations in the form $S(x) = f(x)+S(0)$ and some applications of them.

Finally, in § 3. real-valued additive relations are investigated.

§ 1. The concept of additive relations

Definition 1. 1. Let X and Y be two additively written groupoids.

A relation S with domain X and range contained in Y is called an additive relation from X into Y if for all $a, b \in X$

$$S(a+b) = S(a)+S(b).$$

An additive relation from X into X is simply said to be an additive relation on X.

Example 1. 1. The equality in an additive groupoid is a trivial additive relation on it.

Example 1. 2. Let R be the set of all real numbers. Then the usual ordering on R is an additive relation on R.

Example 1. 3. Let X be a set, and $\mathscr{P}(X)$ be the family of all subsets of X. If the addition in $\mathscr{P}(X)$ is defined by the set union, then the set inclusion in $\mathscr{P}(X)$ is an additive relation on $\mathscr{P}(X)$.

Example 1. 4. If f is an additive function from a groupoid X into a commutative semigroup Y and C is a complex of Y such that $C+C=C$, then the relation S defined by $S(x)=f(x)+C$ is an additive relation from X into Y.

Remark 1. 1. If X, Y are additive groupoids and S is a relation with domain X and range contained in Y, then S is an additive relation from X into Y iff the function φ_S defined by $\varphi_S(x)=S(x)$ for $x\in X$ is an additive function from X into $\mathscr{P}(Y)$, i.e., a holomorphic mapping of X into $\mathscr{P}(Y)$.

Theorem 1. 1. *Let S be an additive relation from a group X into a group Y. Then S is a function iff $S(0)=0$.*

Proof. Obviously, if S is a function, then $S(0)=0$. Conversely, if $S(0)=0$, then from

$$S(x)+S(-x)=S(0)$$

it follows that for each $x\in X$ $S(x)$ has only one element.

Theorem 1. 2. *Let S and T be additive relations from a groupoid X into a commutative semigroup Y. Then the relation $S+T$ defined by*

$$(S+T)(x)=S(x)+T(x)$$

is also an additive relation from X into Y.

Proof. Trivial.

Remark 1. 2. It can be shown that if S_1 and S_2 are additive relations from a module X into a module Y, then

$$T=\{(x_1+x_2,\, y_1+y_2):(x_1,y_1)\in S_1,\ (x_2,y_2)\in S_2\}$$

is also an additive relation from X into Y.

Lemma 1. 1. *Let S be an additive relation from a groupoid X into a groupoid Y. Then for all $A, B\subset X$*

$$S(A+B)=S(A)+S(B).$$

Proof. For $A, B\subset X$ $y\in S(A+B)\Leftrightarrow \exists a\in A$ and $b\in B: y\in S(a+b)\Leftrightarrow \exists a\in A$ and $b\in B: y\in S(a)+S(b)\Leftrightarrow y\in S(A)+S(B)$.

Theorem 1. 3. *Let S be an additive relation from a groupoid X into a groupoid Y, and T be an additive relation from Y into a groupoid Z. Then the composite relation $T\circ S$ is an additive relation from X into Z.*

Proof. This is an immediate consequence of Lemma 1. 1.

§ 2. Representation of additive relations

Definition 2. 1. A function f defined on the domain of a relation S and contained in S is called a choice function or a selection for S.

Remark 2. 1. Note that every choice function for a relation is a choice function for the family of all relation classes of that, and conversely every choice function for a family of sets $\{A_i\}_{i\in I}$ is a choice function for the relation $\{(i, a): i\in I, a\in A_i\}$.

Theorem 2. 1. *Let S be an additive relation from a group X into a group Y. Assume that f is a selection for S. Then for all $x\in X$*

$$f(x)+S(0) \subset S(x) \subset -f(-x)+S(0)$$

and

$$S(0)+f(x) \subset S(x) \subset S(0)-f(-x).$$

PROOF. For all $x\in X$

$$f(x)+S(0) \subset S(x)+S(0) = S(x)$$

and

$$f(-x)+S(x) \subset S(-x)+S(x) = S(0),$$

i.e., $S(x) \subset -f(-x)+S(0)$.

The proof of the other inclusion is quite similar.

Corollary 2. 1. If S is as in Theorem 2. 1., then $S(x)$ has the same cardinality for each $x\in X$.

Definition 2. 2. Let S be an additive relation from a groupoid X with zero into a groupoid Y.

A selection f for S, satisfying at least one of

$$S(x) = f(x)+S(0) \quad \text{and} \quad S(x) = S(0)+f(x)$$

for all $x\in X$, is called a representing selection for S.

A representing selection f for S, satisfying

$$f(x)+S(0) = S(0)+f(x)$$

for all $x\in X$, is said to be normal.

Corollary 2. 2. If S is as in Theorem 2. 1. and f is an odd selection for S, then f is a normal representing selection for S.

Remark 2. 2. If S is as in Theorem 2. 1. and $0\in S(0)$, then there is an odd selection for S.

To prove this, use $S(x)+S(-x) = S(0)$.

Corollary 2. 3. If S is as in theorem 2. 1. and f is an additive selection for S, then f is a normal representing selection for S.

Lemma 2. 1. *Let S be an additive relation from a group X into a group Y. Assume that for some $x_0\in X$ there exists $y_0\in S(x_0)$ such that $-y_0\in S(-x_0)$. Then*

$$S(x_0) = y_0+S(0) = S(0)+y_0.$$

PROOF. The inclusions

$$-y_0+S(x_0) \subset S(-x_0)+S(x_0) = S(0),$$

i.e.,

$$S(x_0) \subset y_0+S(0)$$

and

$$y_0+S(0) \subset S(x_0)+S(0) = S(x_0)$$

imply that $S(x_0) = y_0+S(0)$.

Similarly, from

$$S(x_0)-y_0 \subset S(x_0)+S(-x_0) = S(0),$$

i.e.,

$$S(x_0) \subset S(0)+y_0$$

and

$$S(0)+y_0 \subset S(0)+S(x_0) = S(x_0),$$

it follows that $S(x_0) = S(0)+y_0$.

Theorem 2. 2. *Let S be an additive relation from a group X into a group Y. Assume that f is a selection for S such that $-f(x) \in S(-x)$ for all $x \in X$. Then f is a normal representing selection for S.*

PROOF. This is an immediate consequence of Lemma 2. 1.

Example 2. 1. Let Q be the set of all rational numbers and

$$S = \{(x, y) \in R \times Q : x < y\}.$$

Then S is an additive relation on R, but there is no function f from R into R such that for all $x \in R$ $S(x) = f(x)+S(0)$.

Theorem 2. 3. *Let S be an additive relation from a group X into a group Y. Then every representing selection for S is normal.*

PROOF. Suppose first that f is a selection for S such that for all $x \in X$

$$S(x) = f(x)+S(0).$$

From

$$S(0) = f(0)+S(0)$$

since $f(0) \in S(0)$, it follows that $0 \in S(0)$. Then there is an odd selection g for S. Moreover, by Corollary 2. 2., g is a normal representing selection for S. Thus we have

$$f(x)+S(0) = S(0)+g(x)$$

i.e.,

$$S(0) = -f(x)+S(0)+g(x)$$

for all $x \in X$. Therefore

$$S(-x) = S(0)+g(-x) = -f(x)+S(0)+g(x)+g(-x) = -f(x)+S(0)$$

holds for all $x \in X$. Hence, since $0 \in S(0)$, it follows that $-f(x) \in S(-x)$ for all $x \in X$. Then, by Theorem 2. 2, f is a normal representing selection for S.

A similar proof can be given in the other case.

Theorem 2. 4. *Let S be an additive relation from a group X into a group Y. Then the following propositions are pairwise equivalent:*

(i) $a, -a \in S(0)$ *implies* $a=0$;
(ii) *There exists at most one representing selection for* S;
(iii) *Every representing selection for* S *is additive.*

PROOF. We will prove this theorem by showing successively that (i) implies (ii), (ii) implies (iii), and finally that (iii) implies (i).

Suppose first that (i) is true and assume that f_1 and f_2 are representing selection for S. Then $0 \in S(0)$ and for all $x \in X$

$$f_1(x)+S(0) = f_2(x)+S(0),$$

i.e.

$$S(0) = -f_1(x)+f_2(x)+S(0).$$

Hence, since $0 \in S(0)$, it follows that $-f_1(x)+f_2(x)$ and the opposite of that are in $S(0)$. Consequently, by (i) we obtain $-f_1(x)+f_2(x) = 0$, i.e., $f_1(x)=f_2(x)$ for all $x \in X$.

To show that (ii) implies (iii), let f be a representing selection for S. Then for each $a, b \in X$

$$S(a+b) = f(a+b)+S(0)$$

and

$$S(a+b) = S(a)+S(b) = f(a)+S(0)+f(b)+S(0) =$$

$$= f(a)+f(b)+S(0)+S(0) = f(a)+f(b)+S(0),$$

where we used Theorem 2. 3 too. Hence, by (ii), we can conclude that f is additive.

Finally, to prove that (iii) implies (i), suppose that (iii) is true and $a, -a \in S(0)$. Then $0 \in S(0)$ and, by Lemma 2. 1, we have

$$S(0) = a+S(0).$$

On the other hand, since $0 \in S(0)$, there is a representing selection f for S. Define the function g by

$$g = \begin{cases} f(x) & \text{for} \quad x \neq 0 \\ a & \text{for} \quad x=0. \end{cases}$$

Evidently, g is a representing selection for S. Then by (iii) it follows that g is additive. Thus $g(0)=a=0$, and the proof is complete.

Corollary 2. 4. If S is as in Theorem 2. 4., $0 \in S(0)$ and $a, -a \in S(0)$ implies $a=0$, then there exists a unique representing selection f for S. Moreover, f is additive.

In the sequel we are going to deal with some applications of the results mentioned above.

Theorem 2. 5. *Let X be and additive group and S be a relation with domain X and range contained in X. Then S is an additive equivalence relation on X iff $S(0)$ is a normal subgroup of X and for all $x \in X$*

$$S(x) = x+S(0).$$

PROOF. First, suppose that S is an additive equivalence relation on X. From the additivity and the reflexivity of S, we can infer that $S(0)$ is closed with respect to the addition and $0 \in S(0)$, respectively. Moreover, by Theorem 2. 2., the identity function of X is a normal representing selection for S. To prove that every element of $S(0)$ has its inverse in $S(0)$ let $a \in S(0)$. Then, by the symmetry of S, we can infer that $0 \in S(a)$. Thus from

$$S(a)+S(-a) = S(0),$$

since we also have $-a \in S(-a)$, it follows that $-a \in S(0)$.

For the brevity, the proof of the converse is omitted.

Remark 2. 3. Note that if S is a reflexive additive relation on a group X, then S is transitive.

Remark 2. 4. If S is an additive equivalence relation on a group X, then every selection for S is representing selection for S.

Remark 2. 5. Clearly, S is an additive equivalence relation on a group X iff S is a congruence relation on it.

Example 2. 2. $S = \{(x, y) \in R^2 : y-x \in Q\}$ is an additive equivalence relation on R.

Theorem 2. 6. *Let X be an additive group and S be a relation with domain X and range contained in X. Then S is an additive order relation on X iff $S(0)$ is a non-negativity domain of X and for all $x \in X$*

$$S(x) = x+S(0).$$

PROOF. If S is an additive relation on X, then from the additivity and reflexivity of S, by Theorem 2. 2, it follows that for all $x \in X$

$$S(x) = x+S(0) = S(0)+x.$$

Hence for all $x \in X$ $-x+S(0)+x \subset S(0)$. Moreover, by the additivity of S, we have $S(0)+S(0) \subset S(0)$. To prove $S(0) \cap (-S(0)) = 0$, where $-S(0) = \{x : -x \in S(0)\}$, assume that $a \in S(0)$ and $a \in -S(0)$, i.e., $-a \in S(0)$. Then, by Lemma 2. 1 and the above representation of S, we have

$$S(0) = a+S(0) = S(a).$$

Hence, since $0 \in S(0)$, we can infer that $0 \in S(a)$. Therefore, by the antisymmetry of S, $a \in S(0)$ and $0 \in S(a)$ imply that $a=0$. Finally, to prove $S(0) \cup (-S(0)) = X$, assume that $a \in X$. Then, by the trichotomy of S, we have at least one of $a \in S(0)$ and $0 \in S(a)$. If $a \in S(0)$, then we are ready. If $0 \in S(a)$, then from $S(a) = a+S(0)$ we can conclude $-a \in S(0)$, i.e., $a \in (-S(0))$.

Conversely, if $S(0)$ is a non-negativity domain of X and for all $x \in X$

$$S(x) = x+S(0),$$

then from $S(0) \cap (-S(0)) = 0$ it follows that $0 \in S(0)$. Now, from $S(0)+S(0) \subset S(0)$ we can infer that

$$S(0)+S(0) = S(0).$$

On the other hand, since $-x+S(0)+x \subset S(0)$ for all $x \in X$, we have

$$S(0)+x \subset x+S(0)$$

and

$$x+S(0) \subset S(0)+x,$$

i.e.,

$$x+S(0) = S(0)+x$$

for $x \in X$. Thus for each $a, b \in X$

$$S(a+b) = a+b+S(0) = a+b+S(0)+S(0) =$$
$$= a+S(0)+b+S(0) = S(a)+S(b).$$

Now, it is easy to see that S is an order relation too.

Remark 2. 6. If S is an additive order relation on a group X, then the identity function on X is the unique representing selection for S.

To prove this, use Theorem 2. 4.

Remark 2. 7. Clearly, if X and S are as in Theorem 2. 6, then S is an additive order relation on X iff X with S is an ordered group.

In [1] C. BERGE has defined set-valued linear mappings. Here, we shall define linear relations similarly.

Definition 2. 3. Let X and Y be two vector spaces over the same field F. A relation S with domain X and range contained in Y is called a linear relation from X into Y if for all $a, b \in X$ and $0 \neq \lambda \in F$

(1) $S(a+b) = S(a)+S(b)$,
(2) $0 \in S(0)$,
(3) $S(\lambda a) = \lambda S(a)$.

Example 2. 3. Let $X = \{f: f$ is a function from an interval of R into R having primitives$\}$ and $Y = \{f: f' \in X\}$. Furthermore, define the addition and the scalar multiplication in X and Y on the usual way. Then X and Y are vector spaces over R and the indefinite integral

$$\int = \{(f, F) \in X \times Y : F' = f\}$$

is a linear relation from X into Y.

The following representation theorem exactly characterizes linear relations and is very useful for the investigation of set-valued linear mappings studied by C. BERGE.

Theorem 2. 7. *Let X and Y be two vector spaces over the same field F, and S be a relation with domain X and range contained in Y. Then S is a linear relation from X into Y iff $S(0)$ is a subspace of Y and there exists a linear representing selection for S.*

PROOF. First assume that S is a linear relation from X into Y. From the linearity of S, we have

$$S(0) = S(0)+S(0),$$

$$0 \in S(0)$$

and

$$S(0)=\lambda S(0) \quad \text{for all} \quad 0\neq\lambda\in F.$$

Hence it is obvious that $S(0)$ is a subspace of Y. Let B be a basis for X over F, then for each $x\in X$ there exists a unique function α_x from B into F such that

$$x = \sum_{b\in B} \alpha_x(b)b.$$

Let φ be a choice function for the family $\{S(b)\}_{b\in B}$, and define the function f by

$$f(x) = \sum_{b\in B} \alpha_x(b)\varphi(b).$$

Then $f(b)=\varphi(b)\in S(b)$ for all $b\in B$. Moreover, since $S(-b)=-S(b)$, we also have $-f(b)\in S(-b)$ for all $b\in B$. Thus, by Lemma 2. 1, it follows that for all $b\in B$

$$S(b) = f(b)+S(0).$$

Hence for all $x\in X$

$$S(x) = S\Big(\sum_{b\in B}\alpha_x(b)b\Big) = \sum_{b\in B} S(\alpha_x(b)b) = \sum_{b\in B}\alpha_x(b)S(b) =$$
$$= \sum_{b\in B}\alpha_x(b)f(b)+\sum_{b\in B}\alpha_x(b)S(0) = f(x)+S(0).$$

Finally, it is clear that f is linear.

The proof of the converse is simple and is omitted.

§ 3. Real-valued additive relations

Theorem 3. 1. *Let S be an additive relation from a group X into R. Then exactly one of the following alternatives obtains:*

(A) $\inf S(x)=\sup S(x)$ *is an additive function from X into R;*

(B) $\inf S(x)$ *is an additive function from X into R and* $\sup S(x)=+\infty$ *for all $x\in X$;*

(C) $\inf S(x)=-\infty$ *for all $x\in X$ and* $\sup S(x)$ *is an additive function from X into R;*

(D) $\inf S(x)=-\infty$ *and* $\sup S(x)=+\infty$ *for all $x\in X$.*

Proof. For every $a, b\in X$

$$\inf S(a+b) = \inf S(a)+\inf S(b)$$

and

$$\sup S(a+b) = \sup S(a)+\sup S(b).$$

In particular,

$$\inf S(0) = \inf S(0)+\inf S(0)$$

and

$$\sup S(0) = \sup S(0)+\sup S(0).$$

Hence we have exactly one of the following alternatives:

(a) $\inf S(0) = \sup S(0)=0$;
(b) $\inf S(0) = 0$ and $\sup S(0)=+\infty$;

If (a) prevails, then $S(0)=0$. Thus, by Theorem 1. 1, S is an additive function from X into R.

If we have (b), then from

$$\inf S(0) = \inf S(x)+\inf S(-x)$$

and

$$\sup S(0) = \sup S(x)+\sup S(-x)$$

it is quite obvious that for all $x\in X$

$$\inf S(x) \neq +\infty \quad \text{and} \quad \sup S(x) = +\infty.$$

Thus we have (B).

A similar proof can be given in the case (C) and (D), respectively.

Remark 3. 1. Evidently, if S is as in Theorem 3. 1, then

$$T=\{(x, y):(x, -y)\in S\}$$

is also an additive relation from X into R. Moreover, for all $x\in X$

$$\inf T(x) = -\sup S(x) \quad \text{and} \quad \sup T(x) = -\inf S(x).$$

Thus the investigation of the case (C) can be reduced to that of (B).

Theorem 3. 2. *Let S be an additive relation from a group X into R and f be an additive selection for S. Then the relations S_1 and S_2 defined by*

$$S_1(x)=\{y\in S(x): y\leqq f(x)\}$$

and

$$S_2(x)=\{y\in S(x): y\geqq f(x)\}$$

are additive relations from X into R. Moreover, $S = S_1\cup S_2$ and $f = S_1\cap S_2$.

PROOF. By Corollary 2. 3, we have for all $x\in X$

$$S(x) = f(x)+S(0).$$

Then for all $x\in X$

$$S_1(x) = f(x)+S_1(0) \quad \text{and} \quad S_2(x) = f(x)+S_2(0).$$

Hence it is obvious that S_1 and S_2 are additive.

Remark 3. 2. If S is an additive relation from a group X into R, having additive selection and satisfying (D), then, using Theorem 3. 2 and Remark 3. 1, its investigation can be reduced to that of two additive relations from X into R, having additive selection and satisfying (B).

Theorem 3. 3. *Let S be an additive relation from a group X into R such that for some $x_0\in X$ $\inf S(x_0)\in S(x_0)$. Then $\inf S(x)$ is the unique representing selection for S.*

PROOF. From

$$S(x_0) = S(x_0)+S(0)$$

(c) $\inf S(0)=-\infty$ and $\sup S(0)=0$;
(d) $\inf S(0)=-\infty$ and $\sup S(0)=+\infty$.

it follows that $\inf S(0)=0$ and $0\in S(0)$. Then there exists a selection f for S such that for all $x\in X$

$$S(x)=f(x)+S(0).$$

Hence $\inf S(x)=f(x)$ for all $x\in X$.

Corollary 3. 1. If S is an additive relation from a group X into R satisfying (B) and f is an additive selection for S, then $f(x)=\inf S(x)$ for all $x\in X$.
To prove this, use Corollary 2. 3.

Theorem 3. 4. *Let S be an additive relation from a group X into R. Then 0 is adherent to $S(0)$ in R.*

Proof. By Theorem 3. 1, we may suppose that we have the case (D). Assume indirectly that the conclusion is false. Then for

$$\alpha=\sup\{y\in S(0): y<0\}$$

and

$$\beta=\inf\{y\in S(0): y>0\}$$

we have $\alpha<0<\beta$. Let $\delta=\min\{-\alpha,\beta\}$. Then there are $y_1, y_2\in S(0)$ such that

$$\alpha-\delta<y_1\leqq\alpha \quad\text{and}\quad \beta\leqq y_2<\beta+\delta.$$

Hence

$$\alpha\leqq\alpha+\beta-\delta<y_1+y_2<\alpha+\beta+\delta\leqq\beta.$$

Consequently, $y_1+y_2\notin S(0)$. But this contradicts $S(0)+S(0)=S(0)$, and so the theorem is proved.

Theorem 3. 5. *Let S be an additive relation from a group X into R such that for some $x_0\in X$ $S(x_0)$ has an interior point in R.*

(i) *If S satisfies* (B), *then for all $x\in X$*

$$\inf\{y:[y,+\infty)\subset S(x)\}=\inf S(x)+\inf\{y:[y,+\infty)\subset S(0)\}.$$

(ii) *If S satisfies* (D), *then $S(x)=R$ for all $x\in X$.*

Proof. From

$$S(x_0)+S(-x_0)=S(0)$$

it follows that $S(0)$ has an interior point in R.
Assume first that (B) prevails. Then

$$S(0)+S(0)=S(0)$$

implies that for some $y\in S(0)$ $[y,+\infty)\subset S(0)$. Let

$$\alpha_0=\inf\{y:[y,+\infty)\subset S(0)\}.$$

Then $(\alpha_0,+\infty)\subset S(0)$. Thus, for all $x\in X$

$$(\alpha_0,+\infty)+S(x)\subset S(0)+S(x)=S(x).$$

For $x \in X$, let

$$\alpha_x = \inf \{y : [y, +\infty) \subset S(x)\}.$$

Then for all $x \in X$

$$\alpha_0 + \inf S(x) \geqq \alpha_0 .$$

On the other hand, for all $x \in X$

$$(\alpha_x, +\infty) + S(-x) \subset S(x) + S(-x) = S(0).$$

Hence for all $x \in X$

$$\alpha_x + \inf S(-x) \geqq \alpha_0 ,$$

i.e.,

$$\alpha_x - \inf S(x) \geqq \alpha_0 .$$

Consequently, $\alpha_x = \inf S(x) + \alpha_0$.

Next assume that (D) prevails. Then

$$S(0) + S(0) = S(0)$$

implies that $S(0) = R$. Thus, from

$$S(x) = S(x) + S(0),$$

it follows that $S(x) = R$ for all $x \in X$.

From this theorem, we can easily derive the following results.

Corollary 3. 2. If S is an additive relation on R satisfying (B) and S has an interior point in R^2, then there exists a number c such that for all $x \in X$

$$\inf \{y : [y, +\infty) \subset S(x)\} = cx + \inf \{y : [y, +\infty) \subset S(0)\}.$$

Corollary 3. 3. If S is an additive relation from a group X into R satisfying (B) and for some $x_0 \in X$ $S(x_0)$ is open in R, then for all $x \in X$

$$S(x) = (\inf S(x), +\infty).$$

Corollary 3. 4. If S is an additive relation on R satisfying (B) and S is open in R^2, then there exists a number c such that for all $x \in X$

$$S(x) = (cx, +\infty).$$

Theorem 3. 6. *Let S be an additive relation from a group X into R and denote $\overline{S(x)}$ the closure of $S(x)$ in R. Then*

$$T = \{(x, y) : x \in X, y \in \overline{S(x)}\}$$

is also an additive relation from X into R.

PROOF. Let $a, b \in X$. It is easy to see that

$$\overline{S(a)} + \overline{S(b)} \subset \overline{S(a+b)}.$$

To prove

$$\overline{S(a+b)} \subset \overline{S(a)} + \overline{S(b)}$$

assume first that S does not satisfy (D). If $y \in \overline{S(a+b)}$, then there exists a sequence $y_n \in S(a+b)$ such that $y_n \to y$. Since

$$S(a+b) = S(a)+S(b)$$

there are sequences $\alpha_n \in S(a)$ and $\beta_n \in S(b)$ such that $y_n = \alpha_n + \beta_n$. The assumption implies that α_n and β_n are bounded, so they admit convergent subsequences α_{k_n} and β_{k_n}. Let $\alpha = \lim_{n\to\infty} \alpha_{k_n}$ and $\beta = \lim_{n\to\infty} \beta_{k_n}$. Then $\alpha \in \overline{S(a)}$, $\beta \in \overline{S(b)}$ and $y = \alpha+\beta$, i.e., $y \in \overline{S(a)} + \overline{S(b)}$.

Next assume that S satisfies (D). If $0 \notin S(0)$, then, by Theorem 3. 4, $S(0)$ is dense in $\boldsymbol{R}$, i.e., $\overline{S(0)} = \boldsymbol{R}$. Hence, using

$$\overline{S(x)} + \overline{S(0)} \subset \overline{S(x)}$$

we obtain $\overline{S(x)} = \boldsymbol{R}$ for all $x \in X$. Finally, if $0 \in S(0)$, then there is a representing selection f for S. Thus we have

$$S(a+b) = S(a)+S(b) = S(a)+S(0)+f(b) = S(a)+f(b).$$

Hence it follows

$$\overline{S(a+b)} = \overline{S(a)}+f(b).$$

Since

$$0 \in S(0) = -f(b)+S(b) \subset -f(b)+\overline{S(b)},$$

we can infer that

$$\overline{S(a+b)} \subset \overline{S(a)}+f(b)-f(b)+\overline{S(b)} = \overline{S(a)}+\overline{S(b)}.$$

Corollary 3. 5. If S is as in Theorem 3. 6, then there exists a function f from X into $\boldsymbol{R}$ such that for all $x \in X$

$$\overline{S(x)} = f(x)+\overline{S(0)}.$$

To prove this, observe that $0 \in \overline{S(0)}$.

Corollary 3. 6. If S is as in Theorem 3. 6 and S satisfies (B), then for all $x \in X$

$$\overline{S(x)} = \inf S(x)+\overline{S(0)}.$$

Remark 3. 3. It can be shown that if S is an additive relation on R, then the closure of S in $\boldsymbol{R}^2$ is also an additive relation on R.

Lemma 3. 1. *Let X be a vector space over a field F, B be a basis for X over F, X_b be the subspace of X generated by $b \in B$ and Y be an additively written commutative semigroup.*

Assume that for each $b \in B$ S_b is an additive relation from X_b into Y such that $S_{b_1}(0) = S_{b_2}(0)$ for every $b_1, b_2 \in B$. Then there exists a unique additive relation S from X into Y such that for all $b \in B$ and $x \in X_b$ $S(x) = S_b(x)$.

Proof. Denote α that unique function from $X \times B$ into F such that

$$x = \sum_{b \in B} \alpha(x, b)b.$$

Let a relation S with domain X and range contained in $^*Y$ be defined by

$$S(x) = \sum_{b\in B} S_b(\alpha(x, b)b).$$

Evidently, for all $b \in B$ and $x \in X_b$ $S(x) = S_b(x)$. On the other hand, for each $x_1, x_2 \in X$

$$S(x_1+x_2) = \sum_{b\in B} S_b(\alpha(x_1+x_2, b)b) = \sum_{b\in B} S_b(\alpha(x_1, b)b+\alpha(x_2, b)b) =$$

$$= \sum_{b\in B} S_b(\alpha(x_1, b)b) + \sum_{b\in B} S_b(\alpha(x_2, b)b) = S(x_1)+S(x_2).$$

Finally, to prove the uniqueness, suppose that S^* is also an additive relation from X into Y such that for all $b \in B$ and $x \in X_b$ $S^*(x) = S_b(x)$. Then for each $x \in X$

$$S^*(x) = S^*\big(\sum_{b\in B} \alpha(x, b)b\big) = \sum_{b\in B} S^*(\alpha(x, b)b) = \sum_{b\in B} S_b(\alpha(x, b)b) = S(x).$$

Theorem 3. 7. *There exists an additive relation S on R, satisfying* (D) *such that $0 \in S(0)$, but there is no additive selection for S.*

Proof. Let B be a Hamel basis for R over Q such that $1 \in B$, R_b be the subspace of R generated by $b \in B$ and $s_n = \frac{1}{n!}\sum_{k=0}^{n-1} k!$

Let S_1 be a relation defined by

$$S\left(\frac{m}{n!}\right) = ms_n + Z,$$

where Z is the set of all integers and $m \in Z$. It will be shown that S_1 is an additive relation from $R_1 = Q$ into R. To prove the correctness of the definition of S_1, assume that $\frac{m_1}{n_1!} = \frac{m_2}{n_2!}$ such that $n_1 > n_2$. Then

$$S_1\left(\frac{m_1}{n_1!}\right) = m_1 s_{n_1} + Z = m_2\frac{n_1!}{n_2!}s_{n_1} + Z =$$

$$= m_2 s_{n_2} + m_2\frac{n_2! + \ldots + (n_1-1)!}{n_2!} + Z = m_2 s_{n_2} + Z = S\left(\frac{m_2}{n_2!}\right)$$

For arbitrary $r_1, r_2 \in R_1 = Q$ we can assume no loss of the generality that $r_1 = \frac{m_1}{n!}$ and $r_2 = \frac{m_2}{n!}$. Thus

$$S_1(r_1+r_2) = S_1\left(\frac{m_1+m_2}{n!}\right) = (m_1+m_2)s_n + Z =$$

$$= m_1 s_n + Z + m_2 s_n + Z = S_1(r_1) + S_1(r_2).$$

For $1 \neq b \in B$, let the relation S_b be defined by $S_b(x) = Z$ for $x \in R_b$. Clearly, for all $b \in B$ S_b is an additive relation from R_b into R. Using Lemma 3. 1, let S be that unique additive relation on R such that for all $b \in B$ and $x \in R_b$ $S(x) = S_b(x)$. Then S satisfies (D) and $0 \in S(0)$.

Finally, we shall show that there is no additive selection for S. Assume indirectly that f is an additive selection for S. Then

$$f(1) = n!f\left(\frac{1}{n!}\right) \quad \text{and} \quad f\left(\frac{1}{n!}\right) \in S\left(\frac{1}{n!}\right).$$

Since $0 < s_n < \frac{1}{2}$ if $n > 3$, we have for $n > 3$

$$\inf\left\{|a| : e \in S\left(\frac{1}{n!}\right)\right\} = s_n.$$

Therefore

$$|f(1)| = n!\left|f\left(\frac{1}{n!}\right)\right| \geqq n!\,s_n = 0! + 1! + \ldots + (n-1)!,$$

what is a contradiction.

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