Rearrangement inequalities

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1. *Introduction*

Let $x=(x_1, x_2, \ldots, x_n)$ be an n-tuple of real numbers, where $x_i \in (a, b)$, $i=1, 2, \ldots, n$ and $-\infty \leqq a < b \leqq \infty$. Let us denote by $x^*=(x_1^*, x_2^*, \ldots, x_n^*)$ the same n-tuple rearranged in a nonincreasing order $x_1^* \geqq x_2^* \geqq \ldots \geqq x_n^*$ and $x'=(x_1', x_2', \ldots, x_n')$ the same n-tuple rearranged in a nondecreasing order $x_1' \leqq x_2' \leqq \ldots \leqq x_n'$.

Let $F(x)=F(x_1, x_2, \ldots, x_n)$ $\big(x_i \in (a, b),\ i=1, 2, \ldots, n\big)$ be a real valued function. We say that the function F has the property $\mathscr{P}$ if for any $x=(x_1, x_2, \ldots, x_n)$ $(x_i \in (a, b),\ i=1, 2, \ldots, n)$ the inequality

$$F(x') \leqq F(x) \leqq F(x^*) \tag{1.1}$$

holds.

For a function F with property $\mathscr{P}$ an example is the following one. Suppose that $(a, b)=(0, \infty)$ and $0 \leqq a_1 \leqq a_2 \leqq \ldots \leqq a_n$ is a given sequence. Then the function

$$F(x) = \sum_{i=1}^{n} \frac{a_i}{x_i} \tag{1.2}$$

has property $\mathscr{P}$ (see [1], Theorem 368).

Recently, LONDON [3] proved — as a generalization of (1. 2) — that the functions of the form

$$F(x) = \sum_{i=1}^{n} f\left(\frac{a_i}{x_i}\right) \qquad (0 \leqq a_1 \leqq a_2 \leqq \ldots \leqq a_n) \tag{1.3}$$

have the property $\mathscr{P}$ if $f(t)$ is a convex function for $t \geqq 0$ and satisfies $f(0) \leqq f(t)$ for any $t \geqq 0$. In the same paper one can find — as a result independent from the former one — a sufficient condition for the $\mathscr{P}$ property of the functions of the type

$$F(x) = \sum_{i=1}^{n} f\left(1+\frac{a_i}{x_i}\right) \qquad (0 \leqq a_1 \leqq a_2 \leqq \ldots \leqq a_n). \tag{1.4}$$

This result is a generalization of a theorem given by MINC [4].

In the present note as a common generalization of the earlier results we give a necessary and sufficient condition for the $\mathscr{P}$ property of the functions having the form

$$F(x) = \sum_{i=1}^{n} g_i(x_i), \tag{1.5}$$

where $g_i(t)$, $i=1, 2, \dots, n$, $t \in (a, b)$ are given real valued functions. Further, we also give a sufficient condition for the inequality

$$\sum_{i=1}^{n} \sum_{k=1}^{n} g_{i+k}(x_i y_k) \leqq \sum_{i=1}^{n} \sum_{k=1}^{n} g_{i+k}(x_i^* y_k^*), \tag{1.6}$$

which follows from our main theorem and can be considered as a generalization of a result of WIENER [5] (see also [1], Theorem 386).

2. *The main theorem*

Let $g_i(t)$, $i=1, 2, \dots, n$ be real valued functions defined in the interval (a, b).

Theorem 1. *The function*

$$F(x) = \sum_{i=1}^{n} g_i(x_i) \qquad (x_i \in (a, b), i=1, 2, \dots, n) \tag{2.1}$$

has the property $\mathscr{P}$ if and only if the functions

$$g_k(t) - g_{k+1}(t) \qquad k = 1, 2, \dots, n-1 \tag{2.2}$$

are nondecreasing in the interval (a, b).

PROOF. (i) The condition is necessary. From the property $\mathscr{P}$ it follows

$$\sum_{i=1}^{n} g_i(x_i') \leqq \sum_{i=1}^{n} g_i(x_i) \tag{2.3}$$

for any $x=(x_1, x_2, \dots, x_n)$ $(x_i \in (a, b),\ i=1, 2, \dots, n)$. Let $1 \leqq k \leqq n-1$ and $a < t_1 \leqq \leqq t_2 < b$ be arbitrary variables and let the n-tuple

$$x = (\underset{1}{t_1}, \underset{2}{t_1}, \dots, \underset{k-1}{t_1}, \underset{k}{t_2}, \underset{k+1}{t_1}, \underset{k+2}{t_2}, \dots, \underset{n}{t_2})$$

be substituted into (2. 3). Then we have

$$g_k(t_1) + g_{k+1}(t_2) \leqq g_k(t_2) + g_{k+1}(t_1),$$

i.e. the function $g_k(t) - g_{k+1}(t)$ is nondecreasing.

(ii) The condition is sufficient. First, it should be noted that the function $g_l - g_j$ is nondecreasing if $1 \leqq l < j \leqq n$, since it can be expressed as the sum of nondecreasing functions:

$$g_l - g_j = \sum_{k=l}^{j-1} (g_k - g_{k+1}).$$

This means that the inequality

$$g_l(t) + g_j(s) \leqq g_l(s) + g_j(t) \tag{2.4}$$

is true for all $1 \leqq l < j \leqq n$ and $a < t \leqq s < b$. Let us now consider an arbitrary n-tuple

$$x = (x_1, x_2, \dots, \underset{l}{x_l}, \dots, \underset{j}{x_j}, \dots, x_n) \qquad (1 \leqq l < j \leqq n).$$

We introduce the transformation A_{lj} by

$$A_{lj}x = (x_1, x_2, \dots, \underset{l}{x_j}, \dots, \underset{j}{x_l}, \dots, x_n) \qquad (l < j)$$

which means an interchange of the l-th and j-th elements in x. If $l < j$ and $x_l \geqq x_j$, then by (2. 4) we obtain

$$g_l(x_j) + g_j(x_l) \leqq g_l(x_l) + g_j(x_j),$$

hence

$$(2.5) \quad F(A_{lj}x) = \sum_{\substack{i=1 \\ i \neq l,j}}^{n} g_i(x_i) + g_l(x_j) + g_j(x_l) \leqq \sum_{\substack{i=1 \\ i \neq l,j}}^{n} g_i(x_i) + g_l(x_l) + g_j(x_j) = F(x).$$

The n-tuple x' can be obtained from x by applying a product of transformations of the type A_{lj} $(l < j)$, while $x_l \geqq x_j$, therefore by (2. 5) we have

$$F(x') \leqq F(x).$$

The other inequality can similarly be proved because

$$F(x) \leqq F(A_{lj}x)$$

for $l < j$ and $x_l \leqq x_j$, and x^* can be expressed as consecutive A_{lj} $(l < j)$ transformations performed on x, while $x_l \leqq x_j$. Therefore

$$F(x) \leqq F(x^*).$$

Remark. The factorization used in the proof of the theorem can be verified by induction. The statement is clearly true for $n = 2$. Let us suppose that the statement is also true for $n = k \geqq 2$ and let $x = (x_1, x_2, \dots, x_k, x_{k+1})$ be an arbitrary $(k+1)$-tuple. If $x_1' = x_1$, then the proof is completed, because for the k-tuple $(x_2, x_3, \dots, x_k, x_{k+1})$ the statement is true which implies that it is true for x. If $x_1' = x_j$ $(1 < j)$, then by definition of x' we have $x_1 \geqq x_j$ and thus the $(k+1)$-tuples $A_{1j}x$ and x' will have the same first element. For the further k elements of the $(k+1)$-tuple $A_{1j}x$ the statement will be true by assumption.

For example, if $x = (1, 5, 7, 2, 3, 4, 6)$, then we have

$$x' = A_{67}A_{56}A_{46}A_{35}A_{24}x$$

and similarly

$$x^* = A_{46}A_{37}A_{27}A_{13}x.$$

3. *Special cases*

(a) First, we consider the functions having the form

$$(3.1) \qquad F(x) = \sum_{i=1}^{n} \frac{a_i}{x_i} \qquad (x_i \in (0, \infty),\ i = 1, 2, \dots, n),$$

where $a_i \geqq 0$, $i=1, 2, \ldots, n$ are given numbers. By Theorem 1 the function (3. 1) has the $\mathscr{P}$ property if and only if the functions

$$\frac{a_k}{t}-\frac{a_{k+1}}{t}=\frac{a_k-a_{k+1}}{t} \qquad k=1, 2, \ldots, n-1$$

are nondecreasing, i.e. $a_k \leqq a_{k+1}$, $k=1, 2, \ldots, n-1$. This means that (3. 1) has the property $\mathscr{P}$ if and only if the sequence $\{a_k\}_1^n$ is nondecreasing. This is a generalization of the theorem 368 in [1] because we also proved the necessarity of the condition. We remark that this result is valid for all real, not necessarily nonnegative a_i, $i=1, 2, \ldots, n$.

(b) As a generalization of the example (a) we consider the functions

$$F(x)=\sum_{i=1}^{n} f\left(\frac{a_i}{x_i}\right) \qquad (x_i \in (0, \infty),\ i=1, 2, \ldots, n), \tag{3. 2}$$

where $a_i \geqq 0$, $i=1, 2, \ldots, n$ are given numbers and $f(t)$ is a real valued function in the interval $[0, \infty)$. By Theorem 1 the function (3. 2) has the property $\mathscr{P}$ if and only if the functions

$$\varphi_k(t)=f\left(\frac{a_k}{t}\right)-f\left(\frac{a_{k+1}}{t}\right) \qquad k=1, 2, \ldots, n-1 \tag{3. 3}$$

are nondecreasing in the interval $(0, \infty)$. By reason of this result Theorem 2 of London's paper [3] states the following.

Theorem 2. *If $f(t)$ is convex in the interval $[0, \infty)$, $f(0) \leqq f(t)$ for $t \geqq 0$ and $0 \leqq a_k \leqq$ $\leqq a_{k+1}$, then the function* (3. 3) *is nondecreasing.*

The proof of this theorem will be performed in a way considerable different as given in [3]. For this the following simple Lemma will be used.

Lemma. *If $f(t)$ is convex in the interval $[0, \infty)$ and $f(0) \leqq f(t)$ for $t \geqq 0$, then*

$$f(x_1)+f(x_2) \leqq f(y_1)+f(y_2) \tag{3. 4}$$

for arbitrary $x_i, y_i \in [0, \infty)$ with the property $x_1 \leqq y_1$ and $x_1+x_2 \leqq y_1+y_2$.

Proof. Let $x_3=y_1+y_2-x_1-x_2$ and $y_3=0$. From the well known theorem of Karamata [2] (see also [1], Theorem 108) we have

$$f(x_1)+f(x_2)+f(x_3) \leqq f(y_1)+f(y_2)+f(y_3),$$

hence, because of

$$f(y_3)=f(0) \leqq f(y_1+y_2-x_1-x_2)=f(x_3),$$

we obtain (3. 4).

Proof *of the Theorem 2.* Let $0<x<y$ be arbitrary numbers and we define

$$x_1=\frac{a_k}{x}, \quad x_2=\frac{a_{k+1}}{y}, \quad y_1=\frac{a_{k+1}}{x}, \quad y_2=\frac{a_k}{y}.$$

We have $x_1 \leqq y_1$ and $x_1+x_2 \leqq y_1+y_2$, which implies (3. 4) by the Lemma, i.e.

$$f\left(\frac{a_k}{x}\right)+f\left(\frac{a_{k+1}}{y}\right) \leqq f\left(\frac{a_{k+1}}{x}\right)+f\left(\frac{a_k}{y}\right).$$

Hence

$$\varphi_k(x) \leqq \varphi_k(y).$$

This means that φ_k is nondecreasing.

(c) We now consider the functions

$$F(x) = \sum_{i=1}^{n} \log\left(1+\frac{a_i}{x_i}\right) \qquad \left(x_i \in (0, \infty),\ i=1, 2, \dots, n\right) \tag{3. 5}$$

where $a_i \geqq 0$, $i=1, 2, \dots, n$ are given numbers. By Theorem 1 the function (3. 5) has the property $\mathscr{P}$ if and only if the functions

$$\log\left(1+\frac{a_k}{t}\right) - \log\left(1+\frac{a_{k+1}}{t}\right) = \log\frac{t+a_k}{t+a_{k+1}} \qquad k=1, 2, \dots, n-1$$

are nondecreasing, i.e. the sequence $\{a_k\}_1^n$ is nondecreasing. This result is a generalization of the theorem of MINC [4]. We remark that this result is independent from Theorem 2 because the function $\log(1+t)$ is concave.

(d) As a generalization of the example (c) we consider the functions having the form

$$F(x) = \sum_{i=1}^{n} f\left(1+\frac{a_i}{x_i}\right) \qquad \left(x_i \in (0, \infty),\ i=1, 2, \dots, n\right), \tag{3. 6}$$

where $a_i \geqq 0$, $i=1, 2, \dots, n$ are given numbers and $f(t)$ is a real valued function defined in the interval $[1, \infty)$. By theorem 1 the function (3. 6) has the property $\mathscr{P}$ if and only if the functions

$$\psi_k(t) = f\left(1+\frac{a_k}{t}\right) - f\left(1+\frac{a_{k+1}}{t}\right) \qquad k = 1, 2, \dots, n-1 \tag{3. 7}$$

are nondecreasing in the interval $(0, \infty)$. By reason of this result Theorem 1 of [3] can be presented also in the following form.

Theorem 3. *Let $f(t)$ be a real valued function in the interval $[0, \infty)$ for which*

(i) $f(1) \leqq f(t)$ if $t \geqq 1$; and

(ii) $h(t) = f(e^t)$ is convex in $[0, \infty)$, are valid. Then the function (3. 7) is nondecreasing provided that $0 \leqq a_k \leqq a_{k+1}$.

PROOF. We shall show that this result can easily be obtained from the Lemma, too. By the conditions (i) and (ii) the function $h(t)$ is convex in $[0, \infty)$ and $h(0) \leqq h(t)$ for all $t \geqq 0$. Let $0 < x < y$ be arbitrary numbers and we define

$$x_1 = \log\left(1+\frac{a_k}{x}\right), \quad x_2 = \log\left(1+\frac{a_{k+1}}{y}\right),$$

$$y_1 = \log\left(1+\frac{a_{k+1}}{x}\right), \quad y_2 = \log\left(1+\frac{a_k}{y}\right).$$

We have $x_1 \leqq y_1$ and $x_1 + x_2 \leqq y_1 + y_2$, which — by the Lemma — implies

$$h(x_1) + h(x_2) \leqq h(y_1) + h(y_2),$$

i.e.
$$h\left[\log\left(1+\frac{a_k}{x}\right)\right]-h\left[\log\left(1+\frac{a_{k+1}}{x}\right)\right] \leqq h\left[\log\left(1+\frac{a_k}{y}\right)\right]-h\left[\log\left(1+\frac{a_{k+1}}{y}\right)\right].$$
From this inequality — because of $f(1+t) = h[\log(1+t)]$ — we obtain
$$\psi_k(x) \leqq \psi_k(y),$$
i.e. the function ψ_k is nondecreasing.

4. *On the Wiener's inequality*

The inequality of WIENER [5] states that if $c_2 \geqq c_3 \geqq \ldots \geqq c_{2n} \geqq 0$ and
$$x=(x_1, x_2, \ldots, x_n), \quad y=(y_1, y_2, \ldots, y_n) \quad (x_i \geqq 0,\ y_i \geqq 0,\ i=1, 2, \ldots, n)$$
are arbitrary n-tuples, then

$$\sum_{i=1}^{n}\sum_{k=1}^{n} c_{i+k} x_i y_k \leqq \sum_{i=1}^{n}\sum_{k=1}^{n} c_{i+k} x_i^* y_k^*. \tag{4.1}$$

As a generalization of this theorem we shall prove the following one.

Theorem 4. *Let $g_l(t)$, $l=2, 3, \ldots, 2n$ be real valued functions defined in the interval $(0, \infty)$. For arbitrary n-tuples $x=(x_1, x_2, \ldots, x_n)$ and $y=(y_1, y_2, \ldots, y_n)$ $(x_i, y_i \in (0, \infty),\ i=1, 2, \ldots, n)$ the inequality*

$$\sum_{i=1}^{n}\sum_{k=1}^{n} g_{i+k}(x_i y_k) \leqq \sum_{i=1}^{n}\sum_{k=1}^{n} g_{i+k}(x_i^* y_k^*) \tag{4.2}$$

holds if the functions

$$g_l(t)-g_{l+1}(t) \qquad l=2, 3, \ldots, 2n-1 \tag{4.3}$$

are nondecreasing in the interval $(0, \infty)$.

PROOF. By Theorem 1 we have
$$\sum_{i=1}^{n}\sum_{k=1}^{n} g_{i+k}(x_i y_k) = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} g_{i+k}(x_i y_k)\right) \leqq$$
$$\leqq \sum_{i=1}^{n}\left(\sum_{k=1}^{n} g_{i+k}(x_i y_k^*)\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} g_{i+k}(x_i y_k^*)\right) \leqq \sum_{k=1}^{n}\left(\sum_{i=1}^{n} g_{i+k}(x_i^* y_k^*)\right),$$
which proves the theorem.

As an example we consider the functions $g_l(t)=c_l t$, where c_l, $l=2, 3, \ldots, 2n$ are given numbers. By Theorem 4 the inequality (4. 1) holds if the sequence $\{c_l\}_2^{2n}$ is nonincreasing.

As a further example we define the functions
$$g_l(t) = \frac{c_l}{t}\log\left(1+\frac{c_l}{t}\right) \qquad l=2, 3, \ldots, 2n,$$

where $0 \leqq c_2 \leqq c_3 \leqq \dots \leqq c_{2n}$ are given numbers. By Theorem 4 we have the inequality

$$\sum_{i=1}^{n} \sum_{k=1}^{n} \frac{c_{i+k}}{x_i y_k} \log \left(1 + \frac{c_{i+k}}{x_i y_k}\right) \leqq \sum_{i=1}^{n} \sum_{k=1}^{n} \frac{c_{i+k}}{x_i^* y_k^*} \log \left(1 + \frac{c_{i+k}}{x_i^* y_k^*}\right)$$

for all $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ $(x_i, y_i \in (0, \infty), i = 1, 2, \dots, n)$. This inequality is the same as the following one:

$$G(x, y) = \prod_{i=1}^{n} \prod_{k=1}^{n} \left(\frac{c_{i+k}}{x_i y_k} + 1\right)^{\frac{c_{i+k}}{x_i y_k}} \leqq G(x^*, y^*).$$

References

[1] G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, Inequalities, *Cambridge,* 1934.
[2] T. KARAMATA, Sur une inégalité relative aux functions convexes, *Publ. Math. Univ. Belgrade,* **1** (1932), 145—148.
[3] D. LONDON, Rearrangement inequalities involving convex functions, *Pac. J. Math.,* **34** (1970), 749—753.
[4] H. MINC, Rearrangement inequalities *(in print).*
[5] F. WIENER, Elementarer Beweis eines Reihensatzes von Herrn Hilbert, *Math. Ann.* **68** (1910), 361—366.

(Received July 18, 1972.)