On the Mikusinski operator field (Distributions and operators)

By ANDRÁS BLEYER (Budapest)

Dedicated to the 60th birthday of A. Rapcsák

1. Introduction

Several authors were interested in the connection between distributions and operators. I. FENYŐ dealt with the connection of KOREVAAR fundamental sequences and the operators [3], and the relation between the distributions defined by SIKORSKI and the super functions generated by sequences of operators [2]. Similar investigation can be found is the works of FOIAŞ [4] and of GESZTELYI [6], the latter author dealt with the algebrically integrable operators and showed that every distribution — as an operator — is an algebrically integrable operator. The distributions defined by L. SCHWARTZ with left sided bounded supports are close relation to operators. J. WLOKA gave a necessary and sufficient condition for an operator to be a distribution having left sided bounded support [10].

This note is connected with the above mentioned results. We shall define a continuity property for operators (see 2.) — as a linear transformations of a certain space — which, as we shall show in 3., is equivalent to be condition given by Wloka.

We remark that the topology which is defined in $C^{\infty}_{+,0}$(see in [3]) is finer than the topology of the test function space of distributions. In the second part of the proof of theorem 1 we give a new proof of Wloka form of distribution operators.

In part 4 we shall deal with the τ-invertable operators, with operators which have τ-continuous inverses as reciprocal inverses. Finally, in part 5, we shall investigate some problems which can be solved very easy with operator method, but it seems to be harder to attach in the distribution or classical function sense.

Through the paper we shall use the notations of operators found in the book [9]; and the existence and properties of special operators defined over finite intervals. If we use the pseudo-normed space term we always understand a complete space in uniform structure defined by this semi-norm system.

The purpose of this paper is to give and to apply a characterization of a kind of operators as continuous linear transformations of a pseudo-normed topological space.

2. Definitions and preliminaires

The general form of distributions having left sided bounded supports is

$$T = \sum_{k=1}^{\infty} D^{r_k} f_k(x) \tag{1}$$

where the sequence $\{f_k(x)\}$ is locally finite and the functions f_k are continuous moreover their supports have common lower bound. In what follows we deal with distributions with support contained in $[0, \infty)$, *let us denote by D'_+ their set*, so that this lower bound can be taken equal zero. (See [1] and [10].)

Wloka has shown in [10] that a distribution of the above kind can be corresponded to an operator if and only if it has the form

$$\omega = \sum_{k=0}^{\infty} s^{r_k} e^{-ks} \{\Phi_k(x)\} \tag{2}$$

where the $\{\Phi_k(x)\}$'s are continuous functions on $k < x < \infty$ vanishing on $(-\infty, k)$ and r_k's are positive integers.

In accordance with the purpose of this paper let us define the *space of test functions: $C^{\infty}_{+,0}$* is the space of all infinitely differentiable functions φ for which $\varphi^{(i)}(0)=0$ for every i. The following pseudo-norm system will be introduced to $C^{\infty}_{+,0}$

$$\|f(x)\|_i = \max_{\substack{0 \leq x \leq i \\ 0 \leq j \leq i-1}} |f^{(j)}(x)|, \qquad i = 1, 2, \dots \tag{3}$$

It is evident that the space $C^{\infty}_{+,0}$ with the *pseudo-norm topology τ is a countable normed space with non-decreasing seminorms.*

If M is the Mikusinski field the product $x\{f\}$, $(x \in M, f \in C^{\infty}_{+,0})$ is sensible and $C^{\infty}_{r,0}$ is algebrically isomorphic to a subring of M.

Definition of τ-continuous operator:

$\omega \in M$ is called τ-continuous operator if for every $f \in C^{\infty}_{+,0}$ the product $\omega\{f\} \in C^{\infty}_{+,0}$ and $\omega\{f_n\} \xrightarrow{\tau} \omega\{f\}$ provided that $f_n \xrightarrow{\tau} f$.

3. Characterization of τ-continuous operators and their series forms.

Theorem 1. *An operator $\omega \in M$ is a distribution having a support being in $[0, \infty)$ if and only if ω is a τ-continuous operator.*

PROOF. a) Let ω be a distribution with support being in $[0, \infty)$ then it is of the form (2). We shall show that $\omega\{f\} \in C^{\infty}_{+,0}$ whenever $f \in C^{\infty}_{+,0}$. Since the series (2) converges in the operational sense there is a function $g \in C$ such that*)

$$\sum_{k=0}^{\infty} g e^{-ks} s^{r_k} \{\Phi_k(x)\} \tag{4}$$

*) C is the set of continuous functions defined on $[0, \infty)$.

uniformly converges in every compact interval belonging to $[0, \infty)$. Now since $g\{f\}\in C$ (here $g\{f\}\in C^{\infty}_{+,0}$ also because $f\in C^{\infty}_{+,0}$) we get that the series

$$\sum_{k=0}^{\infty} g e^{-ks} s^{r_k}\{f\}\{\Phi_k(x)\} \tag{4'}$$

also converges uniformly in every interval $[0, \xi]$. It will be shown that this series τ-converges and it's sum is of the form $g\{\varphi\}$ where $\varphi\in C^{\infty}_{+,0}$. Indeed, let $0\leqq x\leqq i$ be fixed then

$$\left\| \sum_{k=k_0}^{k_1} g\{f\} e^{-ks} s^{r_k}\{\Phi_k(x)\} \right\|_i = 0 \tag{5}$$

if $k_0 > i$ because of the properties of the shift operator e^{-ks}. However

$$\sum_{k=0}^{n} g\{f\} e^{-ks} s^{r_k}\{\Phi_k(x)\} \in C^{\infty}_{+,0}$$

and the space $C^{\infty}_{+,0}$ is complete in the τ-topology so the series is τ-convergent too. Nevertheless the partial sums of the series can be written in the form $g\{\varphi_n\}$ where $\varphi_n\in C^{\infty}_{+,0}$ and evidently $g\{\varphi_n\}\xrightarrow{\tau} g\{\varphi\}$ with $\varphi\in C^{\infty}_{+,0}$ thus $g\{\varphi\}=g\omega\{f\}\in C^{\infty}_{+,0}$ which was the required situation. Let $\varphi_n\xrightarrow{\tau}\varphi$ arbitrary in $C^{\infty}_{+,0}$. According to the previous result we have that every function of $C^{\infty}_{+,0}$ multiplies the operator ω to C (in fact to $C^{\infty}_{+,0}$) — compare with the relation (5) — thus we can get

$$\left\| \sum_{k=0}^{\infty} e^{-ks}\{\Phi_k(x)\} s^{r_k}\{\varphi_n(x)\} - \sum_{k=0}^{\infty} e^{-ks}\{\Phi_k(x)\} s^{r_k}\{\varphi(x)\} \right\|_i =$$

$$= \left\| \sum_{k=0}^{k_0} e^{-ks}\{\Phi_k(x)\} s^{r_k}\{\varphi_n(x)\} - \sum_{k=0}^{k_0} e^{-ks}\{\Phi_k(x)\} s^{r_k}\{\varphi(x)\} \right\|_i \leqq$$

$$\leqq \sum_{k=0}^{k_0} \| e^{-ks}\{\Phi_k(x)\} s^{r_k}\{\varphi_n(x)-\varphi(x)\}\|_i =$$

$$= \sum_{k=0}^{k_0} \Big(\max_{\substack{0\leqq x\leqq i\\ 0\leqq \nu\leqq i-1}} \Big| \int_0^x \Phi_k(x-t+k)\big(\varphi_n^{(\nu+r_k)}(t)-\varphi^{(\nu+r_k)}(t)\big)\,dt \Big|\Big) \leqq$$

$$\leqq \sum_{k=0}^{k_0} \Big(\max_{\substack{0\leqq x\leqq i\\ 0\leqq \nu\leqq i-1}} |\varphi_n^{(\nu+r_k)}(x)-\varphi^{(\nu+r_k)}(x)|\, i\big(\max_{0\leqq t\leqq x} |\Phi_k(x-t+k)|\big)\Big) < \varepsilon$$

Here we used the continuity of the functions $\Phi_k(x)$ moreover the τ-continuity of s^{r_k}, for every r_k, from which

$$\varphi_n^{(\nu)}(x) \xrightarrow{\tau} \varphi^{(\nu)}(x)$$

follows whenever $\varphi_n\xrightarrow{\tau}\varphi$.

Thus we proved that every operator of the form (2), i.e. every distribution of D'_+, is τ-continuous.

b) In the proof of the converse we need the following theorem:

Theorem 2. *$(X; \Sigma)$ and $(X'; \Sigma')$ are countably normed spaces, are complete in topologies τ_Σ resp. $\tau'_{\Sigma'}$: $A: X \to X'$ is a linear transformation. A is continuous if and only if there are indexes i_j and positive constants C_{i_j} such that*

$$\|Ax\|_i \leqq C_{i_j} \|x\|_{i_j} \tag{6}$$

for every $x \in X$.

(See in [7] and [5].)

We may remark that $i_j > i_{j-1}$ may be assumed by non-decreasing semi-norms.

Let ω be a τ-continuous operator. *It will be shown that ω is of the form* (2).

If an operator q is equivalent to continuous function at t let us denote by $q(t)$ the value of this function at t. Set $[\omega\{\varphi\}](t) = \eta(t)$ $0 \leqq t \leqq j$, t is fixed. For each t $\eta(t)$ can be considered to be a linear functional acting on the set of functions from $C^\infty_{+,0}$ whose domains are restricted to $[0, i_j]$.

We have chosen the interval $[0, i_j]$ since

$$\|\omega\{\varphi\}\|_i \leqq C_{i_j} \|\varphi\|_{i_j} \tag{7}$$

holds by Theorem 2, and by (7)

$$|[\omega\{\varphi\}](t)| \leqq C_{i_j} \|\varphi\|_{i_j} \tag{8}$$

follows. This means that $\{[\omega\{\ \}](t)\}$ is a continuous linear functional on the set of functions from $C^\infty_{+,0}$ considering their domains only on $[0, i_j]$. Denote the closure (the complition) of this space by H_j in the norm topology $\|\cdot\|_{i_j}$. Evidently, by an argument based upon (8), the norm conditions are satisfied by $\|\{[\omega\{\ \}](t)\}\|_j = \inf C_{i_j}(t)$ hence $\{[\omega\{\ \}](t)\}$ can be extended to the whole space H_j with norm preserving. This fact holds for each $t \in [0, j]$ so thus for every fixed $\varphi \in H_j$ is a real or complex valued function acting on $0 \leqq t \leqq j$. (Evidently, $(i_j - 1)$-times differentiable functions for which $\varphi^{(\nu)}(0) = 0$ $(\nu = 0, 1, \dots, i_j - 1)$ are elements of H_j.) Let us consider the (i_j+1) power of the operator of integration.*)

$$l^{i_j+1} = \left\{\frac{t^{i_j}}{i_j!}\right\} \tag{9}$$

As a consequence of the above mentioned fact $l^{i_j+1} \in H_j$. Assign $f_j(t) = [\omega l^{i_j+1}](t)$. We shall show that the functions $f_j(t)$ are of bounded variation on $[0, j]$. For this let us choose an arbitrary division of $[0, j]$:

$$0 = t_0 < t_1 < \dots < t_n = j$$

*) $l^{i_j+1} \notin C^\infty_{+,0}$, it is the cause of the necessity of the extension of $[\omega\{\ \}](t)$.

and by denoting $\xi_m = j - t_m$

$$\sum_{m=1}^{n} |f_j(t_m) - f_j(t_{m-1})| = \sum_{m=1}^{n} |e^{-\xi_m s} f_j(j) - e^{-\xi_{m-1} s} f_j(j)| =$$

$$= \sum_{m=1}^{n} |[(e^{-\xi_m s} l^{i_j+1} - e^{-\xi_{m-1} s} l^{i_j+1})\omega](j)| \leqq$$

$$\leqq \sum_{m=1}^{n} c_{ij} \|\omega\|_j \|e^{-\xi_m s} l^{i_j+1} - e^{-\xi_{m-1} s} l^{i_j+1}\|_{ij} \leqq$$

$$\leqq c_{ij} \|\omega\|_j \left(\max_{\substack{0 \leqq t \leqq i_j \\ 0 \leqq \nu \leqq i_j - 1}} \sum_{m=1}^{n} \left| \frac{(t-\xi_m)^{i_j-\nu}}{(i_j-\nu)!} - \frac{(t-\xi_{m-1})^{i_j-\nu}}{(i_j-\nu)!} \right| \right) \leqq \text{**)}$$

$$\leqq c_{ij} \|\omega\|_j \max_{0 \leqq \nu \leqq i_j - 1} \left(\operatorname{var} \left(\frac{t^{i_j-\nu}}{(i_j-\nu)!} \right) \right).$$

Similarly as in $(+)$ we have that

$$l^{i_j+1}\{[\omega\{\varphi\}](t)\} = \{[\omega l^{i_j+1}](t)\}\{\varphi\}$$

for all $\varphi \in C^{\infty}_{+,0}$ on $0 \leqq t \leqq j$. Thus

$$l^{i_j+1}\{[\omega\{\varphi\}](t)\} = \{[\omega\{\varphi\}](t\} l^{i_j+1} = \{\varphi\}\{[\omega l^{i_j+1}](t)\}$$

i.e.

(10) $$\{[\omega\{\varphi\}](t)\} = s^{i_j+2}\{f_j^*\}\{\varphi\}$$

on $0 \leqq t \leqq j$, where $\{f_j^*\} = l\{f_j\}$ is a continuous function on $0 \leqq t \leqq j$. (Here it was used the fact $f_j(0) = 0$.) Since

$$\|\omega\{\varphi\} - s^{i_j+2}\{f_j^*\}\{\varphi\}\|_k = 0$$

whenever $j > k$, we have

(11) $$\omega\{\varphi\} = \tau - \lim_{j \to \infty} s^{i_j+2}\{f_j^*\}\{\varphi\}.$$

Thus from the relation $s^{i_j+2}\{f_j^*\}\{\varphi\} - s^{i_{j-1}+2}\{f_{j-1}^*\}\{\varphi\} = 0$ on $0 \leqq t \leqq j-1$ it follows

(12) $$\{f_j^*\}\{\varphi\} - s^{i_{j-1}-i_j}\{f_{j-1}^*\}\{\varphi\} = 0$$

on $0 \leqq t \leqq j-1$.

Now by (11) and (12)

$$\omega\{\varphi\} = \tau - \lim_{j \to \infty} s^2[s^{i_1}\{f_1^*\}\{\varphi\} + s^{i_2}(\{f_2^*\}\{\varphi\} - l^{i_2-i_1}\{f_1^*\}\{\varphi\}) + \cdots$$

$$\cdots + s^{i_j}(\{f_j^*\}\{\varphi) - l^{i_j-i_{j-1}}\{f_{j-1}^*\}\{\varphi\}) + \cdots] =$$

$$= \sum_{j=1}^{\infty} s^{i_j+2} e^{-(j-1)s}\{\Phi_k(t)\}\{\varphi(t)\}.$$

Hence ω is an operator of form (2). Q.e.d.

**) Here it was used that the shift operator $e^{-\lambda s}$ $(\lambda > 0)$ is existing as an operator defined on a finite interval and is commutable with every continuous functional $[\omega\{\ \}](t)$.

4. On τ-invertable operators.

It is easy to see that not every τ-continuous operator has an τ-continuous inverse. Indeed, let $\omega = \frac{\{\varphi\}}{\{f\}}$ where $\varphi \in C^{\infty}_{+,0}$ and $f \notin C^{\infty}_{+,0}$ and let ω be τ-continuous, then $\omega^{-1} = \frac{\{f\}}{\{\varphi\}}$ is not a τ-continuous operator. (For example let $f = l$ and φ be arbitrary in $C^{\infty}_{+,0}$.)

Let $\omega = \frac{\{f\}}{\{g\}}$ be a τ-continuous operator then the equation $f\eta = \varphi g$, which is of the Volterra type, always has a solution φ of $C^{\infty}_{+,0}$ for each $\eta \in C^{\infty}_{+,0}$.

For ω^{-1} to be a τ-continuous operator a necessary condition is that the equation:

$$f\eta = g\varphi \tag{13}$$

is solvable in $C^{\infty}_{+,0}$ for every $\varphi \in C^{\infty}_{+,0}$.

Is this condition (13) sufficient?

It is easy to prove that the *condition* (13) *is also sufficient whenever* $\inf |f(x)| = A > 0$. Indeed, if $f\eta_n - f\eta = g\varphi_n - g\varphi$, then, if η_n, η, φ_n and φ are of $C^{\infty}_{+,0}$

$$s^{\nu} f\eta_n - s^{\nu} f\eta = \int_0^t f(x)\big(\eta_n^{(\nu)}(t-x) - \eta^{(\nu)}(t-x)\big)\,dx =$$

$$= \int_0^t g(x)\big(\varphi_n^{(\nu)}(t-x) - \varphi^{(\nu)}(t-x)\big)\,dx = s^{\nu} g\varphi_n - s^{\nu} g\varphi$$

so by continuity of f and g we have

$$f(x_t) \int_0^t \big(\eta_n^{(\nu)}(t-x) - \eta^{(\nu)}(t-x)\big)\,dx = g(x_t') \int_0^t \big(\varphi_n^{(\nu)}(t-x) - \varphi^{(\nu)}(t-x)\big)\,dx$$

i.e.

$$\|\eta_n - \eta\|_i \leqq \frac{\max\limits_{0 \leqq x \leqq i} |g(x)|}{\min\limits_{0 \leqq x \leqq i} |f(x)|} \|\varphi_n - \varphi\|_i \to 0 \qquad (n \to \infty)$$

It is imaginable if $f(x) \to 0$ slowly enough then (13) is not sufficient. (We have not yet any examples for this.)

The following is a simple fact.

Theorem 3. *The set of all τ-continuous operators having τ-continuous inverses is a multiplicative subgroup of M.*

Proof. Indeed, denote the above set by M_τ, for $\frac{f}{g} \in M_\tau$, $\left(\frac{f}{g}\right)^{-1} = \frac{g}{f} \in M_\tau$ also; if $\frac{f_i}{g_i} \in M_\tau$ $(i = 1, 2)$ then evidently $\frac{f_1}{g_1} \cdot \frac{f_2}{g_2} \in M_\tau$ follows from $\psi = \frac{f_1}{g_1}\varphi \in C^{\infty}_{+,0}$ and $\frac{f_2}{g_2}\psi \in C^{\infty}_{+,0}$ and from their continuity. Q.e.d.

Let us consider some examples to τ-invertable operators. Let $\{f(x)\}=\{x^{\nu}\beta(x)\}$, $(\beta(+0)\neq 0,\ \nu>0)$ then the equation (13) always has a continuous solutions. (See in [3].) Moreover the equation $f\eta = l^r\{gs^r\varphi\}$ has a solution of n-times differentiable functions if r is suitably large. But $\varphi^{(\nu)}(0)=0$ for all ν thus the new equations are equal to (13), hence η is infinitely differentiable solution of (13). It is known that if $g=\{x^{\mu}\alpha(x)\}$, $(\mu>0,\ \alpha(+0)\neq 0)$ then $f/g \leftrightarrow T\in D'_+$, so in our term f/g is a τ-continuous operator, hence

$$\frac{f}{g}=\frac{\{x^{\nu}\beta(x)\}}{\{x^{\mu}\alpha(x)\}}\in M_\tau.$$

(See 5 b.)

5. Some remarks on the space $C^{\infty}_{+,0}$

5. a) *Let ω be a τ-continuous operator then the equation*

$$\omega\varphi-\varphi_0=\varphi \qquad (0\not\equiv\varphi_0(x)\in C^{\infty}_{+,0}) \tag{14}$$

— which is called generalized inhomogen Volterra type equation — has a solution in $C^{\infty}_{+,0}$ if and only if $\varphi_0\in(\omega-1)C^{\infty}_{+,0}$.

Indeed, in the case $\varphi_0\in(\omega-1)C^{\infty}_{+,0}$, $\varphi_0=(\omega-1)\varphi_1$ however φ_1 satisfies (14). If φ_1 is a solution of (14) then $\varphi_0=(\omega-1)\varphi_1$ and 1 is τ-continuons thus $\varphi_0\in(\omega-1)C^{\infty}_{+,0}$.

5. b) *If we let out the τ-continuity of ω but assume that ω^{-1} is a locally integrable function, then* (14) *has only one solution of $C^{\infty}_{+,0}$ for every non zero $\varphi_0\in C^{\infty}_{+,0}$.*

By the formal solution

$$\varphi_1=\frac{\varphi_0}{\omega-1}$$

we can get

$$\varphi_1=\frac{1}{\omega}\varphi_0\left(\frac{1}{1-\dfrac{1}{\omega}}\right)=\frac{1}{\omega}\varphi_0\left(\sum_{k=0}^{\infty}\left(\frac{1}{\omega}\right)^k\right)=\frac{\varphi_0}{\omega}\left(1+\{k(t)\}\right)$$

thus $\varphi_1\in C^{\infty}_{+,0}$ because $1\in M_\tau$; ω^{-1} and $\{k(t)\}$ are locally integrable functions. Thus we can see if $\omega=p^{-1}$, inverse of a local integrable function p, then $(\omega-1)^{-1}$ is τ-continuous.

References

[1] I. FENYŐ, Disztribuciók és operátorok (Distributions and oprators), I—III. *Mat. Lapok.* **14** (1963), 213—245, **15** (1964), 24—52, **16** (1965), 309—338.
[2] I. FENYŐ, A Mikusinszki-féle operator fogalom és a disztribuciók fogalma közötti kapcsolatról. (On the connection between the operators and distributions) *MTA Osztály-közlemények* **8** (1958), 385—392.
[3] I. FENYŐ, Über den Zusammenhang zwischen den Mikusinskischen Operatoren und Distributionen, *Math Nachr.* **19** (1958), 161—164.
[4] C. FOIAŞ, Approximation des operateurs de J. Mikusinski par des functions continues, *Studia Math.* **21** (1961), 73—74.
[5] I. M. GELFAND, G. E. SCHILOV, Verallgemeinerte Funktionen II. *Berlin* 1962.
[6] E. GESZTELYI, Anwendung der Operatorrechnung auf lineare Differentialgleichungen mit Polinom-Koefficienten, *Publ. Math. (Debrecen)* **10** (1963), 215—243.
[7] J. HORVÁTH, Topological vector spaces and distributions I. *London* 1966.
[8] I. MIKUSINSKI, Le calcul operationne d'intervalle fini, *Studia Math.* **15** (1956), 225—251.
[9] I. MIKUSINSKI, Operational Calculus, *New York*, 1959.
[10] J. WLOKA, Distributionen und Operatoren. *Math. Ann.* **140** (1960), 227—244.

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