On the behaviour of Laplace-integrals under conformal mappings of the region of convergence

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Let ζ be a complex number with $0<|\zeta|<1$ and by

$$\psi(z) = \frac{z-\zeta}{1-\bar{\zeta}z}$$

denote a conformal mapping of the unit circle onto itself. TURÁN [9] proved

Theorem A: *Given any ζ with $0<|\zeta|<1$ there is a function*

$$f_1(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$$

regular for $|z|<1$ with convergent $\sum_{\nu=0}^{\infty} a_\nu$ and such that the series

$$f_2(z) = f_1(\psi(z)) = \sum_{\nu=0}^{\infty} b_\nu(\zeta) z^\nu$$

diverges for the corresponding

$$z = \psi^{-1}(1) = \frac{1+\zeta}{1+\bar{\zeta}}.$$

If $f_1(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ is convergent at $z=1$ then $f_2(z)$ is Abel-summable at $z=\psi^{-1}(1)$.

Several authors investigated similar problems (see e.g. the references in [5]). Especially ALPÁR [1] proved

Theorem B: *Given any ζ with $0<|\zeta|<1$ there is a function*

$$f_1(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$$

regular for $|z|<1$ *with* $\sum_{\nu=0}^{\infty}|a_\nu|<\infty$ *and such that*

$$f_2(z) = f_1(\psi(z)) = \sum_{\nu=0}^{\infty} b_\nu(\zeta) z^\nu$$

with

$$\sum_{\nu=0}^{\infty} |b_\nu(\zeta)| = \infty.$$

In this note we shall prove some similar results for *Laplace-integrals*. Let α, β, γ, δ be real numbers with

$$\alpha\delta - \beta\gamma = 1. \tag{1}$$

Then the right halfplane $\operatorname{Re} s > 0$ is mapped conformally onto itself by

$$\varphi(s) = \frac{\alpha s + i\beta}{\delta - i\gamma s}.$$

In the following we consider nontrivial conformal mappings, i.e. we assume

$$\alpha\gamma \neq 0. \tag{2}$$

Consider s with $\operatorname{Re} s > 0$ and set

$$f_1(s) = \int_0^\infty e^{-st} F_1(t)\,dt + K_1.$$

Then in general neither of the following relations hold for $\operatorname{Re} s > 0$:

$$f_2(s) = f_1(\varphi(s)) = \int_0^\infty e^{-st} F_2(t)\,dt + K_2$$

$$f_2(s) = f_1(\varphi(s)) = \int_0^\infty e^{-st}\,dF_2(t).$$

As an example take ($\operatorname{Re} s > 0$)

$$f_1(s) = \log\frac{s-i}{i(s-2i)} = \int_0^\infty e^{-st}\frac{e^{2it}-e^{it}}{t}\,dt - \frac{i\pi}{2}$$

and

$$f_2(s) = f_1\left(\frac{2s+i}{1-is}\right) = \log s.$$

(See [4; vol. 1, p. 251, ex. 12].) But as a positive result we have

Theorem 1. *Given a function* $f_1(s)$ *for* $\operatorname{Re} s > 0$ *with the representation*

$$f_1(s) = \int_0^\infty e^{-st} F(t)\,dt$$

such that the following integrals exist:

(i) $$\Phi(t) = \int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau)\sqrt{\tau}\, J_1\left(\frac{2}{\gamma}\sqrt{t\tau}\right) d\tau \quad \text{for} \quad t > 0$$

(ii) $$\int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau)\, d\tau$$

where $J_1(z)$ denotes the Besselfunction

$$J_1(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu \left(\frac{z}{2}\right)^{2\nu+1}}{\nu!\,(\nu+1)!}.$$

Then

$$f_2(s) = f_1(\varphi(s)) = \int_0^\infty e^{-st} e^{-\frac{i\delta t}{\gamma}} t^{-\frac{1}{2}} \Phi(t)\, dt + \int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau)\, d\tau.$$

The analogue to Theorem B is

Theorem 2. *Let α, β, γ, δ satisfy* (1) *and* (2). *Then there exists a function $f_1(s)$ having for* $\operatorname{Re} s > 0$ *the form*

$$f_1(s) = \int_0^\infty e^{-st} F_1(t)\, dt + K_3$$

with

$$\int_0^\infty |F_1(t)|\, dt < \infty,$$

but such that

$$f_2(s) = f_1(\varphi(s)) = \int_0^\infty e^{-st} F_2(t)\, dt + K_4$$

with

$$\int_0^\infty |F_2(t)|\, dt = \infty.$$

Finally we obtain an analogue of the second part of theorem A.

Theorem 3. *Suppose $k \geqq 0$ and let α, β, γ, δ satisfy* (1) *and* (2). *Assume for* $\operatorname{Re} s > 0$ *the existence of*

$$f_{1,k}(s) = \lim_{x\to\infty} \frac{1}{x^k} \int_0^x (x-t)^k e^{-st}\, dF_1(t) = \mathscr{L}^k(F_1)^{*})$$

$$f_{2,k}(s) = f_{1,k}(\varphi(s)) = \mathscr{L}^k(F_2).$$

*) For this notation see [3, p. 314], [6, p. 4].

a) *Then the existence of* $f_{1,k}(s)$ *at* $s=0$ *implies*

$$\lim_{r\to 0+} f_{2,k}\left(-\frac{i\beta}{\alpha}+r\right) = \lim_{r\to 0+} f_{1,k}(r),$$

i.e. if $f_{1,k}(0)$ *exists then* $f_{2,k}(s)$ *is Abel-summable at* $s=\varphi^{-1}(0)=-\dfrac{i\beta}{\alpha}$.

b) *If* $f_{1,k}(s)$ *is bounded in a Stolz-region, i.e. for s with*

$$|\arg s| < \psi < \frac{\pi}{2}, \qquad |s| < 1,$$

then the existence of $\lim\limits_{r\to 0+} f_{1,k}(r)$ *implies*

$$\lim_{r\to 0+} f_{2,k}\left(-\frac{i\beta}{\alpha}+r\right) = \lim_{r\to 0+} f_{1,k}(r),$$

i.e. if $f_{1,k}(s)$ *is Abel-summable at* $s=0$ *then the same is true for* $f_{2,k}(s)$ *at* $s=\varphi^{-1}(0)=$ $=-\dfrac{i\beta}{\alpha}$.

Proof of Theorem 1: It is known [4; vol. 1, p. 245, ex. 36] that for $a\geqq 0$ and $\operatorname{Re} s>0$

$$1-e^{-\frac{a}{s}} = \int_0^\infty e^{-st}\sqrt{\frac{a}{t}}\, J_1(2\sqrt{at})\, dt$$

hence

$$1-\exp\left(-\frac{\tau}{\gamma}\frac{1}{\gamma s+i\delta}\right) = \frac{1}{\gamma}\int_0^\infty e^{-st}e^{-\frac{i\delta t}{\gamma}}\sqrt{\frac{\tau}{t}}\, J_1\left(\frac{2}{\gamma}\sqrt{\tau t}\right) dt.$$

Theorem 1 is proved if in the following repeated integrals the change in the order of integration may be justified. Since (i) and (ii) we have

$$\int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau)\, d\tau - \int_0^\infty e^{-\varphi(s)\tau} F(\tau)\, d\tau = \int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau)\left\{1-\exp\left(-\frac{\tau}{\gamma}\frac{1}{\gamma s+i\delta}\right)\right\} d\tau =$$

$$= \frac{1}{\gamma}\int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau)\int_0^\infty e^{-ts}e^{-\frac{i\delta\tau}{\gamma}}\sqrt{\frac{\tau}{t}}\, J_1\left(\frac{2}{\gamma}\sqrt{\tau t}\right) dt\, d\tau =$$

$$= \frac{1}{\gamma}\int_0^\infty e^{-st}e^{-\frac{i\delta t}{\gamma}}\, t^{-\frac{1}{2}}\int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F(\tau)\sqrt{\tau}\, J_1\left(\frac{2}{\gamma}\sqrt{\tau t}\right) d\tau\, dt.$$

The proof of the change in the order of integration is similar to the proof of

$$\int_0^\infty e^{-st}\sqrt{t}\int_0^\infty J_1(2\sqrt{\tau t})G(\tau)\, d\tau\, dt = \frac{1}{s^2}\int_0^\infty e^{-\frac{t}{s}}\sqrt{t}\, G(t)\, dt.$$

For this see [3, p. 133] or [8].

PROOF OF THEOREM 2: Theorem B was proved by connecting the coefficients a_n and $b_n(\zeta)$ by a non-absolute-regular matrix method. We cannot generalize this proof since in general $f_1(\varphi(s))$ is no Laplace-integral.

We prove theorem 2 by giving an example of functions satisfying the conditions of theorem 1.

Let $\alpha, \beta, \gamma, \delta$ satisfy (1) and (2) and define $F_1(t)$ by

$$F_1(t) = \begin{cases} e^{\frac{i\alpha t}{\gamma}} \dfrac{1}{t\sqrt{t-1}} & t > 1 \\ 0 & t \leqq 1 \end{cases}.$$

It is known that [4; vol. 2, p. 18, ex. 5]

$$\int_0^\infty e^{-\frac{i\alpha\tau}{\gamma}} F_1(\tau)\sqrt{\tau} J_1\left(\frac{2}{\gamma}\sqrt{\tau t}\right) d\tau = \gamma \cdot t^{-\frac{1}{2}} \cdot \sin\frac{2}{\gamma}\sqrt{t}$$

and [4; vol. 1, p. 136, ex. 26] for $\operatorname{Re} s \geqq 0$

$$f_1(s) = \pi \operatorname{Erf} c\left(\sqrt{s - \frac{i\alpha}{\gamma}}\right) = \int_0^\infty e^{-st} F_1(t)\, dt = \pi - 2\sqrt{\pi} \int_0^{\sqrt{s-\frac{i\alpha}{\gamma}}} e^{-t^2}\, dt.$$

From Theorem 1 and [4; vol. 1, p. 154, ex. 34] we deduce

$$F_2(t) = -e^{-\frac{i\delta t}{\gamma}} \cdot \gamma \cdot t^{-t} \sin\frac{2}{\gamma}\sqrt{t}$$

and for $\operatorname{Re} s \geqq 0,\ s \neq -i\delta/\gamma$

$$\pi - f_2(s) = \pi \operatorname{Erf}\left(\frac{1}{\gamma} \frac{1}{\sqrt{s + \frac{i\delta}{\gamma}}}\right) = \int_0^\infty e^{-st} F_2(t)\, dt = 2\sqrt{\pi} \int_0^{\frac{1}{\gamma}\frac{1}{\sqrt{s+\frac{i\delta}{\gamma}}}} e^{-t^2}\, dt.$$

Hence we have

$$\int_0^\infty |F_1(t)|\, dt = \int_1^\infty \frac{1}{t\sqrt{t-1}}\, dt < \infty$$

and

$$\int_0^\infty |F_2(t)|\, dt = \int_0^\infty \frac{\left|\sin\frac{2}{\gamma}\sqrt{t}\right|}{t}\, dt = \infty.$$

PROOF OF THEOREM 3: Let $r > 0$. We have $f_{2,k}\left(r - \frac{i\beta}{\alpha}\right) = f_{1,k}\left(\frac{\alpha^2 r}{1 - i\alpha\gamma r}\right)$ and for $\operatorname{Re} s > 0$ the functions $f_{1,k}(s)$, $f_{2,k}(s)$ are holomorphic (see [3, p. 330], [7, p. 284]). $\frac{\alpha^2 r}{1 - i\alpha\gamma r}$ approaches 0 inside an angle $< \pi$ with vertex at 0 so that the generalized

Abel-limitation theorem can be applied [3, p. 331], [7, p. 284] in case a). By proving case *b* in a similar way we use theorem I of [2, p. 457].

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