Special multiplicative deviations

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Dedicated to the 60th birthday of A. Rapcsák

1. Introduction

Let $K(+, \cdot)$ be a field, $K_0 = K \setminus \{0\}$ and let $(A, +)$ be an abelian group.

We say, that the function $M: K_0^2 \to A$ is *a multiplicative deviation* if there exist a function $m: K_0 \to A$, such that

$$(1) \qquad M(x, y) = m(x) + m(y) - m(xy)$$

for all $x, y \in K_0$. The function m is called the generator function of M. It is obvious, that if m and m' are generator functions of M, then $\mu = m - m'$ is a homomorphism of $(K_0, \cdot)$ into $(A, +)$, that is

$$(2) \qquad \mu(xy) = \mu(x) + \mu(y)$$

for all $x, y \in K_0$.

In this paper we will deal with following problems:

A. What conditions are sufficient, that the function

$$(3) \qquad F(x, y) = f(x+y) - f(xy+1), \quad f: K \to A, \quad x, y \in K_0$$

will be a multiplicative deviation?

B. Find the functions $f: R \to R$; $g, h: R_0 \to R$ satisfying

$$(4) \qquad f(x+y) + g(xy) = h(x) + h(y)$$

for all $x, y \in R_0$, where R is the field of real numbers and $R_0 = R \setminus \{0\}$.

C. Find the functions $f: R \to R$; $g, h: R_1 \to R$ satisfying

$$(5) \qquad f(x+y) + g(x+y-xy) = h(x) + h(y)$$

for all $x, y \in R_1$, where R_1 denotes the set $R \setminus \{1\}$.

The results of problem B. are a generalization of results of I. ECSEDI (see [3]).

2. Problem A.

Theorem 1. *Let $K(+,\cdot)$ be a field of characteristic $\neq 2$, $K_0 = K\setminus\{0\}$ and let $(A,+)$ be an abelian group* in which every equation $2z=a(a\in A)$ has a unique solution $z\in A$. *If the function $F:K_0\to A$, defined by* (3), *is a multiplicative deviatation, then the function f has the form*

$$f(x) = l\left[\left(\frac{x}{2}\right)^2\right] + B(x)+C, \tag{6}$$

where the function $l:K\to A$ satisfies the functional equation

$$l\left[\left(\frac{x+y}{2}\right)^2\right] - l\left[\left(\frac{x-y}{2}\right)^2\right] = l(xy); \qquad x, y\in K_0, \tag{7}$$

the function $B:K_0\to A$ satisfies the functional equation

$$B(x+y) = B(x)+B(y), \qquad x, y\in K \tag{8}$$

and $C\in(A,+)$ is a constant.

Proof. If the function $F:K_0^2\to A$, defined by (3), is a multiplicative deviation, then

$$F(-x,-y) = f[-(x+y)]-f(xy+1) = m(-x)+m(-y)-m(xy)$$

and therefore

$$f(x+y)-f[-(x+y)] = m(x)-m(-x)+m(y)-m(-y).$$

From this last equation we obtain that the function

$$\bar{f}(x) = f(x)-f(-x) \tag{9}$$

and the function

$$\bar{m}(x) = m(x)-m(-x)$$

satisfy the functional equation

$$\bar{f}(x+y) = \bar{m}(x)+\bar{m}(y); \qquad x, y\in K_0 \tag{10}$$

and $\bar{f}$, $\bar{m}$ are odd functions. By

$$M_1(x,y) = \bar{f}(x+y)-\bar{f}(xy+1) = \bar{m}(x)+\bar{m}(y)-\bar{m}(xy)-\bar{m}(1),$$

M_1 is a multiplicative deviation, therefore the function

$$M^*(x,y) = f^*(x+y)-f^*(xy+1) \tag{11}$$

is a multiplicative deviation too, where f^* defined by

$$f^*(x) = f(x)+f(-x) = 2f(x)-\bar{f}(x) \tag{12}$$

and f^* is even function.

By (12) it follows

$$f(x)=\frac{f^*(x)+\bar{f}(x)}{2}, \tag{13}$$

thus it is sufficient to determine the functions f^* and $\bar{f}$.

Setting $x=y$ in (10) we obtain the identity $f(2x)=2\overline{m}(x)$, and putting this in (10) we get

$$2\bar{f}(x+y)=\bar{f}(2x)+\bar{f}(2y); \qquad x, y\in K_0. \tag{14}$$

On the other hand it follows, by (14), that

$$2\bar{f}(x)=2\bar{f}\left(\frac{x-y}{2}+\frac{x+y}{2}\right)=\bar{f}(x-y)+\bar{f}(x+y)=$$

$$=\frac{1}{2}[\bar{f}(2x)+\bar{f}(-2y)+\bar{f}(2x)+\bar{f}(2y)]=\bar{f}(2x),$$

because $\bar{f}$ is an odd function. From this and (14) we obtain

$$\bar{f}(x)=B(x), \tag{15}$$

where B satisfies the functional equation (8).

The function M^*, defined by (11), is a multiplicative deviation and f^* is an even function, this reason it follows that

$$M^*(-x,-y)=f^*(x+y)-f^*(xy+1)=M^*(x,y)$$

and therefore the generator function of M^* satisfies the identity

$$m^*(x)-m^*(-x)=C_1,$$

where $C_1\in(A,+)$ constante, so it follows by

$$M^*(x,-y)=m^*(x)+m^*(-y)-m^*(-xy)=m^*(x)+m^*(y)-m^*(xy)$$

that the function f^* satisfies the functional equation

$$f^*(x+y)-f^*(xy+1)=f^*(x-y)-f^*(-xy+1),$$

which can be written in the form

$$f^*(x+y)-f^*(x-y)=f^*(xy+1)-f^*(-xy+1)=l(xy). \tag{16}$$

By setting $x=y\to\frac{x}{2}$ in (16), it follows

$$f^*(x)=f^*(0)+l\left[\left(\frac{x}{2}\right)^2\right], \tag{17}$$

which give with (16), that the function l satisfies the functional equation (7).

From (15) and (17) it follows by (13), that

$$f(x) = \frac{1}{2} l\left[\left(\frac{x}{2}\right)^2\right] + \frac{1}{2} B(x) + \frac{1}{2} f^*(0).$$

If $f(x)$ is a solution of this problem then $2f(x)$ is a solution too and vica versa, that is f has the form (6) and $C = f^*(0) \in (A, +)$.

3. Problem B.

Theorem 2. *If the functions f, g, h satisfy the functional equation* (4) *for all $x, y \in R_0$, then we have*

$$f(x) = l\left[\left(\frac{x}{2}\right)^2\right] + B(x) + C, \tag{18}$$

$$h(x) = l\left[\left(\frac{x}{2}\right)^2\right] + B(x) + \varphi(x) + C', \tag{19}$$

$$g(x) = h(x) - f(x+1) + h(1), \tag{20}$$

where the functions l and B satisfy the functional equation (8) *for all $x, y \in R$, the function φ satisfies the functional equation*

$$\varphi(xy) = \varphi(x) + \varphi(y); \qquad x, y \in R_0, \tag{21}$$

$$C' = h(1) - l\left(\frac{1}{4}\right) - B(1),\ C \text{ is an arbitrary constant.}$$

Proof. Setting $y = 1$ in (4) we obtain

$$f(x+1) + g(x) = h(x) + h(1). \tag{22}$$

Putting $x \to xy$ in (22) and subtracting this from (4), we get the functional equation

$$f(x+y) - f(xy+1) = h(x) + h(y) - h(xy) - h(1), \tag{23}$$

thus the function

$$F(x, y) = f(x+y) - f(xy+1)$$

is a multiplicative deviation on R_0.

Using Theorem 1., we obtain, that the function f has the form (18), where $l: R \to R$ satisfies (7) for all $x, y \in R_0$ and $B: R \to R$ satisfies (8) for all $x, y \in R$.

Now we will prove, that l satisfies (8) for all $x, y \in R$.

By the substitution

$$\begin{cases} xy = u \\ \left(\dfrac{x-y}{2}\right)^2 = v \end{cases} \tag{24}$$

we obtain from (7), that the function l is additive for all

$$(u, v) \in D = \{(u, v) | v > 0, \; v > -u\},$$

that is on the domain (u, v) of transformation (24) (see figure 1.).

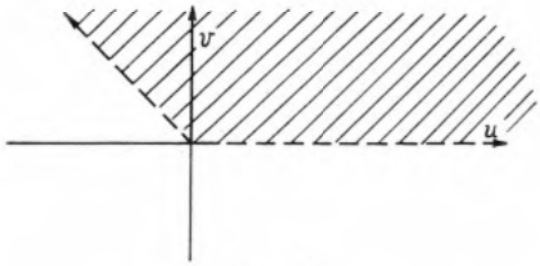

Fig. 1

It is easy to see (similarly then in [4] and [5]), that l is additive for all $(u, v) \in \in R \times R$.

If l is additive for some (u, v), then l is additive for all

$$(u, v) \in D_1 = \{(u, v) | v > -u\}$$

(see figure 2.).

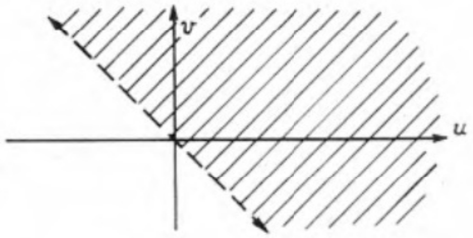

Fig. 2

If $(u, v) \notin D_1$, then there exist a real number $x > 0$, that the points (x, u), $(x+u, v)$, $(x, u+v)$ are elements of D_1, because the inequalities system $x > 0$, $x+u+v > 0$ has a solution x for any $(u, v) \in D_1$.

In this case there are

$$l(u) = l(x+u) - l(x),$$

$$l(v) = l(x+u+v) - l(x+u),$$

$$l(u+v) = l(x+u+v) - l(x).$$

It follows from these equations that l is additive for all $(u, v) \in R \times R$.

Setting (18) in (23), we obtain the equation

$$l\left[\left(\frac{x}{2}\right)^2\right] + l\left[\left(\frac{y}{2}\right)^2\right] - l\left[\left(\frac{xy}{2}\right)^2\right] + B(x) + B(y) - B(xy) + C' = h(x) + h(y) - h(xy)$$

for all $x, y \in R_0$. We can see that the function

$$\varphi(x) = h(x) - l\left[\left(\frac{x}{2}\right)^2\right] - B(x) - C'$$

satisfies the functional equation (21), from which it follows (19) for $h(x)$. From (22) we obtain (20).

Thus the Theorem 2. is proved.

4. Problem C.

Theorem 3. *If the functions f, g, h satisfy the functional equation* (5) *for all $x, y \in R_1$, then we have*

$$\text{(25)} \qquad \begin{cases} f(x) = l\left[\left(\frac{2-x}{2}\right)^2\right] + B(2-x) + C, \\ h(x) = l\left[\left(\frac{1-x}{2}\right)^2\right] + B(1-x) + \varphi(1-x) + C', \\ g(x) = h(x) - f(x) + h(0), \end{cases}$$

where the functions l and B satisfy the functional equation (8) *for all $x, y \in R$ and $\varphi(x)$ satisfies* (21). *C and C' are arbitrary constants with*

$$C' = h(0) - l\left(\frac{1}{4}\right) - B(1).$$

Proof. By substitution $x \to 1-x$; $y \to 1-y$ in (4) it follows the functional equation

$$f(2-x-y) + g(1-xy) = h(1-x) + h(1-y); \qquad x, y \in R_0,$$

which implies the functional equation (4) for $\tilde{f}$, $\tilde{g}$, $\tilde{h}$ with

$$\text{(26)} \qquad \begin{cases} \tilde{f}(x) = f(2-x) \\ \tilde{g}(x) = g(1-x) \\ \tilde{h}(x) = h(1-x), \end{cases}$$

From Theorem 2. and (26) we obtain (25).

5. Remarks and problems

1. If the functions f, g, h satisfy the functional equation (4) for all $x, y \in R$, then we obtain from Theorem 2., that

$$f(x) = l\left[\left(\frac{x}{2}\right)^2\right] + B(x) + C,$$

$$h(x) = l\left[\left(\frac{x}{2}\right)^2\right] + B(x) + C',$$

$$g(x) = h(x) - f(x+1) + h(1),$$

where the functions l and B satisfy the functional equation (8) for all $x, y \in R$. C, C' and $h(1)$ are arbitrary constants with $C' = h(1) - l\left(\frac{1}{4}\right) - B(1)$, because in this case $\varphi(x) \equiv 0$ (see [1]).

2. We can take a similarly remark for equation (5), if (5) is true for all $x, y \in R$.

3. By solving certain functional equations it have an important role other multiplicative deviations, therefore it would have been intrasting the research of other deviations.

4. What is the general solution of functional equation (4) for all $x, y \in R_+ = \{x | x > 0\}$? (Continuous solutions see in [2]).

Reference

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