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The existence of maximal elements and equilibria in Frechet spaces

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Abstract. In this paper we first give some existence theorems of maximal elements of condensing correspondences. Then two existence theorems of equilibria of abstract economies (resp., generalized games) are derived by maximal element theorems in Frechet spaces. Finally, a fixed point theorem is proved which improves the corresponding results of Barbolla (1985), Gale and Mas-Colell (1975) and Florenzano (1981).

1. Introduction

Let E be a vector space and $A \subset E$. We shall denote by $\operatorname{co} A$ the convex hull of A. If A is a subset of a topological space X, the interior and closure of A in X are denoted by $\operatorname{int}_X A$ and $\operatorname{cl}_X A$, respectively; or simply $\operatorname{int} A$ and $\operatorname{cl} A$ if there is no ambiguity. Let X be a non-empty set. We shall denote by 2^X the family of all subsets of X. Let X and Y be sets and $F, G : X \to 2^Y$ a set-valued mapping. Then (i): the graph of F denoted by Graph$F$, is the set $\{(x,y) \in X \times Y : y \in F(x)\}$; and (ii): the mapping $F \cap G : X \to 2^Y$ is defined by $(F \cap G)(x) := F(x) \cap G(x)$ for each $x \in X$. Suppose X and Y are topological spaces and $F : X \to 2^Y$. Then (1): F is said to be lower (resp., upper) semicontinuous if for any closed (res., open)

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subset U of Y, the set $\{x \in X : F(x) \subset U\}$ is closed (resp., open) in X; (2): F has open lower sections if $F^{-1}(y) := \{x \in X : y \in F(x)\}$ is open in X for each $y \in Y$; (3): F is said to be continuous if F is both upper and lower semicontinuous; and (4): F has a maximal element if there exists a point $x \in X$ such that $F(x) = \emptyset$.

If X is a set, Y is a subset of a vector space and $F : X \to 2^Y$ such that for each $x \in X$, $\operatorname{co} F(x) \subset Y$, the mapping $\operatorname{co} F : X \to 2^Y$ is defined by $(\operatorname{co} F)(x) := \operatorname{co} F(x)$ for each $x \in X$. If $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ are collections of sets and $F_i : \Pi_{j\in I} X_j \to 2^{Y_i}$ is a set-valued mapping for each $i \in I$, the mapping $\Pi_{i\in I} F_i : \Pi_{i\in I} X_i \to 2^{\Pi_{i\in I} Y_i}$ is defined by $(\Pi_{i\in I} F_i)(x) := \Pi_{i\in I} F_i(x)$ for each $x \in \Pi_{i\in I} X_i$. We note that if X is a topological space, Y is a topological vector space and $F : X \to 2^Y$ is lower semicontinuous, then the mapping $\operatorname{co} F$ is also lower semicontinuous by Proposition 2.6 of Michael [17].

Let I be a (finite or infinite) set of agents (resp., players). An abstract economy (resp., a generalized game) $\mathcal{G} = (X_i; A_i; P_i)_{i\in I}$ is defined as a family of triples $(X_i; A_i; P_i)_{i\in I}$, where X_i is a topological space, where for each $i \in I$, $A_i : \Pi_{j\in I} X_j \to 2^{X_i}$ is a constraint correspondence and $P_i : \Pi_{j\in} X_j \to 2^{X_i}$ is a preference correspondence. An equilibrium point for $\mathcal{G}$ (e.g., see Borglin and Keiding [3, p. 315], Tarafdar [24, p. 212] or Yannelis and Prabhakar [26, p. 242]) is a point $x^* \in X := \Pi_{j\in I} X_j$ such that for each $i \in I$, $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$ where $\pi_i : X \to X_i$ is the projection for each $i \in I$.

Throughout this paper, C denotes a lattice with a least element zero 0. Now we recall some definitions (e.g., see Fitzpatrick and Petryshyn [7]).

Let X be a Hausdorff topological vector space. Then a mapping $\Psi : 2^X \to C$ is said to be a measure of non-compactness provided that the following conditions hold for any $A, B \in 2^X$:

(1) $\Psi(A) = 0$ if and only if A is precompact.

(2) $\Psi(\overline{\operatorname{co}} A) = \Psi(A)$, where $\overline{\operatorname{co}} A$ denotes the closed convex hull of A.

(3) $\Psi(A \cup B) = \max\{\Psi(A), \Psi(B)\}$.

It follows from (3) that if $A \subset B$, then $\Psi(A) \le \Psi(B)$. The above notion is a generalization of the set-measure of non-compactness introduced by Kuratowski [13] and the ball-measure of non-compactness introduced by Gohberg et al [11] defined either in terms of a family of seminorms when X is a locally convex topological vector space or of a single norm when X is a Banach space. For more details we refer the readers to Fitzpatrick and Petryshyn [7].

Let $\Psi : 2^X \to C$ be a measure of non-compactness of X and $D \subset X$. A mapping $T : D \to 2^X$ is said to be Ψ-condensing provided that if $\Omega \subset D$ and $\Psi(T(\Omega)) \geq \Psi(\Omega)$, then Ω is relatively compact.

Note that if $T : D \to 2^X$ is a compact mapping (i.e., $T(D) = \bigcup_{x \in D} T(x)$ is precompact), then T is Ψ-condening for any measure of non-compactness Ψ. Various Ψ-condensing mappings which are not compact have been considered by Borisovich et al [4], Massatt [14], Nussbaum [18], Petryshyn and Fitzpatrick [19], Reich [20], Sadovskii [21] and many others. Moreover, when the measure of non-compactness Ψ is either the set-measure of non-compactness or ball-measure of non-compactness, Ψ-condensing mappings are called condensing mappings, e.g., see Massatt [14], Nussbaum [18] and Sadovskii [21].

In this paper we first give some existence theorems of maximal elements of Ψ-condensing correspondences. Two existence theorems of equilibria of abstract economies are then proved in Frechet spaces. Finally some fixed point theorems are also derived. These results improve corresponding results of Barbolla [2], Gale and Mas-Colell [10] and Mehta [15–16].

2. Maximal elements

Recall that a Frechet space is a locally convex Hausdorff topological vector space whose topology is induced by a complete translation invariant metric.

We first state the following result which is an easy consequence of Theorem 1 of Fitzpatrick and Petryshyn [7, p. 18].

Lemma 2.1. *Let D be a non-empty closed and convex subset of a Frechet space X and $\Psi : 2^X \to C$ be a measure of non-compactness. Suppose a multivalued correspondence $F : D \to 2^D$ is upper semicontinuous and Ψ-condensing with non-empty compact convex values. Then F has a fixed point.*

Now we give the following results on the existence of maximal elements for Ψ-condensing correspondences:

Theorem 2.2. *Let E be a Frechet space, $\Psi : 2^X \to C$ be a measure of non-compactness and X a non-empty closed and convex subset of E. Suppose that $P : X \to 2^X$ is a multivalued mapping such that the following conditions are satisfied:*

(i) *for each $x \in X$, $x \notin \operatorname{co} P(x)$;*

(ii) *for each $x \in X$ such that $P(x) \neq \emptyset$, there exists $y \in X$ such that $x \in \operatorname{int} P^{-1}(y)$;*

(iii) *P is Ψ-condensing.*

Then P has a maximal element in X.

Proof. Suppose that the conclusion were false. Then for each $x \in X$, $P(x) \neq \emptyset$. Now define the mapping $F : X \to 2^X$ by $F(x) = \{y \in X : x \in \operatorname{int} P^{-1}(y)\}$ for each $x \in X$. Then for each $y \in X$, $F^{-1}(y) = \operatorname{int} P^{-1}(y)$ is open in X so that F has open lower sections. Also for each $x \in X$, $F(x)$ is non-empty by (ii). Hence the mapping $\operatorname{co} F : X \to 2^X$ also has open lower sections by Lemma 5.1 of Yannelis and Prabhakar [26, p. 239]. Now by Browder's selection theorem in [5] (see also Theorem 3.1 of [26, p. 235]), there exists a continuous (single-valued) function $f : X \to X$ such that $f(x) \in \operatorname{co} F(x)$ for each $x \in X$. Since P is Ψ-condensing, so is the mapping $\operatorname{co} P$. Thus $\operatorname{co} F$ is Ψ-condensing since $\operatorname{co} F(x) \subset \operatorname{co} P(x)$ and hence f is also Ψ-condensing. Now f satisfies all hypotheses of Lemma 2.1. By Lemma 2.1, f has a fixed point $x_0 \in X$, i.e. $x_0 = f(x_0) \in \operatorname{co} F(x_0) \subset \operatorname{co} P(x_0)$, which contradicting the condition (i). Therefore the conclusion must hold. □

Theorem 2.2 improves Theorem of Mehta [15, p. 70] in the following ways: (1) X is a subset of a Frechet space instead of a Banach space; (2) X need not bounded; and (3) P is Ψ-condensing instead of being set condensing.

We also have the following existence theorem of maximal elements for a lower semicontinuous correspondence which has closed and convex values.

Theorem 2.3. *Let E be a Frechet space and X a non-empty closed and convex subset of E. Suppose that $P : X \to 2^X$ is lower semicontinuous with closed and convex values such that the following conditions are satisfied:*

(i) *for each $x \in X$, $x \notin P(x)$;*

(ii) *P is Ψ-condensing.*

Then P has a maximal element in X.

Proof. Suppose that the conclusion were false, then $P(x)$ is non-empty closed and convex subset of X for each $x \in X$. Since P is also lower semicontinuous, by Theorem 3.2" of Michael [17], there exists a continuous (single-valued) function $f : X \to X$ such that $f(x) \in P(x)$ for each $x \in X$. Since P is Ψ-condensing, so is f. Now f satisfies all hypotheses of Lemma 2.1. By Lemma 2.1, there exists a point $x_0 \in X$ such that $x_0 = f(x_0) \in P(x_0)$, which also contradicts the condition (i). Therefore there exists $x \in X$ such that $P(x) = \emptyset$. $\square$

We note that Theorem 2.3 deals with the existence of maximal elements for lower semicontinuous correspondences. In what follows, we shall give the following existence theorem for upper semicontinuous correspondences.

Theorem 2.4. *Let E be a Frechet space, $\Psi : 2^E \to C$ be a measure of non-compactness and X a non-empty closed and convex subset of E. Suppose that $P : X \to 2^X$ is upper semicontinuous and Ψ-condensing with compact convex values. If P is irreflexive (i.e, for each $x \in X$, $x \notin P(x)$), then there exists $x_0 \in X$ such that $P(x_0) = \emptyset$.*

Proof. If the conclusion were false, then $P(x)$ is non-empty closed and convex subset of X for each $x \in X$. Thus by Lemma 2.1, there exists $x_0 \in X$ such that $x_0 \in P(x_0)$, but this contradicts the assumption that P is irreflexive. Therefore the conclusion must hold. $\square$

3. Existence of equilibria

In this section, we shall prove the existence of equilibria for abstract economies. Before we prove our main results, we will need Lemma 2.10 in [23] which is a generalization of Lemma 6.1 in [26, p. 241]; for completeness, we shall include its proof:

Lemma 3.1. *Let X and Y be topological spaces, A be a closed (resp., open) subset of X. Suppose $F_1 : X \to 2^Y$ and $F_2 : A \to 2^Y$ are lower (resp., upper) semicontinuous such that $F_2(x) \subset F_1(x)$ for all $x \in A$. Then the mapping $F : X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_1(x), & \text{if } x \notin A, \\ F_2(x), & \text{if } x \in A \end{cases}$$

is also lower (resp., upper) semicontinuous.

Proof. Let U be any closed (resp., open) subset of Y. Clearly we have

$$\begin{aligned}\{x \in X : F(x) \subset U\} &= \{x \in A : F_2(x) \subset U\} \cup \{x \in X \setminus A : F_1(x) \subset U\} \\ &= \{x \in A : F_2(x) \subset U\} \cup \{x \in X : F_1(x) \subset U\}.\end{aligned}$$

Since A and U are closed (resp., open) and F_1 and F_2 are lower (resp., upper) semicontinuous, the set $\{x \in X : F(x) \subset U\}$ is also closed (resp., open). Therefore, F is lower (resp., upper) semicontinuous. □

In the rest part of this section, the set I of agents (resp., players) is assumed to be countable.

Theorem 3.2. *Let $\Gamma = (X_i; A_i; P_i)_{i \in I}$ be an abstract economy where I is countable. Suppose for each $i \in I$, the following conditions are satisfied:*

(i) *X_i is a non-empty closed convex subset of a Frechet space E_i;*

(ii) *A_i is upper semicontinuous with non-empty compact convex values;*

(iii) *the mapping $A : X \to 2^X$ defined by $A(x) = \Pi_{i \in I} A_i(x)$ for each $x \in X = \Pi_{i \in I} X_i$ is Ψ-condensing, where $\Psi : 2^{\Pi_{j \in I} E_j} \to C$ is a measure of non-compactness;*

(iv) *for each $x \in X$, $\pi_i(x) \notin A_i(x) \cap P_i(x)$;*

(v) *the set $U_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X;*

(vi) *P_i is upper semicontinuous on U_i such that for each $x \in U_i$, $P_i(x)$ is closed and convex.*

Then there exists $x^* \in X$ such that for each $i \in I$, $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P(x^*) = \emptyset$.

Proof. Fix an $i \in I$. Define $\psi_i : U_i \to 2^{X_i}$ by $\psi_i(x) = A_i(x) \cap P_i(x)$ for each $x \in U_i$, then ψ_i is upper semicontinuous on U_i by (ii), (vi) and Proposition 2.5.2 of [1, p. 71]. We now define the correspondence $F_i : X \to 2^{X_i}$ by

$$F_i(x) = \begin{cases} \psi_i(x), & \text{if } x \in U_i, \\ A_i(x), & \text{if } x \notin U_i. \end{cases}$$

By Lemma 3.1, F_i is upper semicontinuous with non-empty compact convex values.

Now let $F(x) = \Pi_{i\in I}F_i(x)$ for each $x \in X$. Then F is a upper semicontinuous with non-empty compact convex values by Theorem 7.3.14 of [12, p. 88]. Since $F(x) \subset A(x)$ for each $x \in X$ and A is Ψ-condensing, F is also Ψ-condensing. Since I is countable, X is also a closed and convex subset of the Frechet space $\Pi_{i\in I}E_i$. Therefore F satisfies all hypotheses of Lemma 2.1. By Lemma 2.1, there exists $x^* \in X$ such that $x^* \in F(x^*)$. By (iv), $x^* \notin U_i$ for each $i \in I$; it follows that for each $i \in I$, $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P(x^*) = \emptyset$. □

Theorem 3.3. *Let $\Gamma = (X_i; A_i; P_i)_{i\in I}$ be an abstract economy where I is countable. Suppose for each $i \in I$, the following conditions are satisfied:*

(i) *X_i is a non-empty closed and convex subset of a Frechet space E_i;*

(ii) *A_i is lower semicontinuous with non-empty closed convex values;*

(iii) *the mapping $A : X \to 2^X$ defined by $A(x) = \Pi_{i\in I}A_i(x)$ is Ψ-condensing for each $x \in X = \Pi_{i\in I}X_i$, where $\Psi : 2^{\Pi_{j\in I}E_j} \to C$ is a measure of non-compactness;*

(iv) *for each $x \in X$, $\pi_i(x) \notin A_i(x) \cap P_i(x)$;*

(v) *the set $U_i := \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is closed in X.*

(vi) *the mapping $A_i \cap P_i$ is lower semicontinuous on U_i such that for each $x \in U_i$, $A_i(x) \cap P_i(x)$ is closed and convex.*

Then there exists $x^* \in X$ such that for each $i \in I$, $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

Proof. Fix an $i \in I$. Define $F_i : X \to 2^{X_i}$ by

$$F_i(x) = \begin{cases} A_i(x) \cap P_i(x), & \text{if } x \in U_i, \\ A_i(x), & \text{if } x \notin U_i. \end{cases}$$

By Lemma 3.1, F_i is lower semicontinuous with non-empty closed and convex values. Then by Michael's selection theorem in [17, Theorem 3.2''] again, there exists a continuous (single-valued) mapping $f_i : X \to X_i$ such that $f_i(x) \in F_i(x)$ for each $x \in X$.

Now define $f : X \to X$ by $f(x) = \{f_i(x)\}_{i\in I}$ for each $x \in X$. Then f is continuous and $f(x) \in F(x) = \Pi_{i\in I}F_i(x) \subset \Pi_{i\in I}A_i(x)$. Since A is Ψ-condensing, it follows that f is also Ψ-condensing. Since I is countable, $X = \Pi_{i\in I}X_i$ is a non-empty closed and convex subset of the Frechet space $\Pi_{i\in I}E_i$. Therefore f satisfies all hypotheses of Lemma 2.1. By Lemma 2.1, there exists $x^* \in X$ such that $f(x^*) = x^*$. Note that for each $i \in I$, if

$x_i^* \in U_i$, then $\pi_i(x^*) = f_i(x^*) \in A_i(x^*) \cap P_i(x^*)$ which contradicts (iv). Hence for each $i \in I$, we must have $\pi_i(x^*) \notin U_i$ and thus $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$. □

We also have that:

Theorem 3.4. *Let $\Gamma = (X_i; A_i; P_i)_{i\in I}$ be an abstract economy where I is countable. Suppose for each $i \in I$, the following conditions are satisfied:*

(i) *X_i is a non-empty closed and convex subset of a Frechet space E_i;*

(ii) *A_i is continuous with non-empty compact and convex values;*

(iii) *the mapping $A: X = \Pi_{i\in I} X_i \to 2^X$ defined by $A(x) = \Pi_{i\in I} A_i(x)$ for each $x \in X$ is Ψ-condensing, where $\Psi : 2^{\Pi_{j\in I} E_j} \to C$ is a measure of non-compactness;*

(iv) *the set $U_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is either open or closed in X;*

(v) *the mapping $A_i \cap P_i$ is lower semicontinuous on U_i such that for each $x \in U_i$, $A_i(x) \cap P_i(x)$ is closed and convex.*

Then there exists $x^* \in X$ such that for each $i \in I$, either $\pi_i(x^*) \in A_i(x^*) \cap P_i(x^*)$ or $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

Proof. Fix an $i \in I$. By (v), Theorem 3.2" of Michael [17, p. 367], it follows that there exists a single-valued continuous mapping $f_i : U_i \to X_i$ such that such that $f_i(x) \in A_i(x) \cap P_i(x)$ for each $x \in U_i$. We are going to prove the results by following two cases.

Case 1. Suppose U_i is open in X. Define $F_i : X \to 2^{X_i}$ by

$$F_i(x) = \begin{cases} \{f_i(x)\}, & \text{if } x \in U_i, \\ A_i(x), & \text{if } x \notin U_i; \end{cases}$$

then by (ii) and Lemma 3.1, F_i is upper semicontinuous with non-empty compact convex values.

Case 2. Suppose U_i is closed. Define $F_i' : X \to 2^{X_i}$ by

$$F_i'(x) = \begin{cases} \{f_i(x)\}, & \text{if } x \in U_i, \\ A_i(x), & \text{if } x \notin U_i; \end{cases}$$

then by (ii) and Lemma 3.1, F_i' is lower semicontinuous with non-empty compact convex values. By the extension result of continuous selections of Michael [17, Proposition 1.4(b)], there exists another single-valued continuous mapping $f_i' : X \to X_i$ such that $f_i'(x) \in F_i'(x)$ for each $x \in X$ (and indeed $f_i'(x) = f_i(x)$ if $x \in U_i$). Let $F_i : X \to 2^{X_i}$ be defined by $F_i(x) := \{f_i'(x)\}$ for each $x \in X$. Then F_i is upper semicontinuous (in fact, continuous) with non-empty compact convex values.

Now we define a set-valued mapping $F\colon X \to 2^X$ by $F(x) := \Pi_{i\in I} F_i(x)$ for each $x \in X$. Then F is upper semicontinuous with non-empty compact convex values (e.g., see Theorem 7.3.14 of Klein and Thompson [12, p. 88]) and $F(x) \subset A(x)$ for each $x \in X$. Since A is Ψ-condensing, so is the mapping F. By Lemma 2.1, there exists $x^* \in X$ such that $x^* \in F(x^*)$. It follows that for each $i \in I$, either $\pi_i(x^*) \in A_i(x^*) \cap P_i(x^*)$ or $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$. □

By Theorem 3.1''' of Michael [17, p. 368] instead of his Theorem 3.2'' in [17, p. 367], the same argument used in proving Theorem 3.4 can likewise prove the following:

Theorem 3.5. *Let $\Gamma = (X_i; A_i; P_i)_{i\in I}$ be an abstract economy where I is countable. Suppose for each $i \in I$, the following conditions are satisfied:*

(i) *X_i is a non-empty closed and convex subset of a separable Banach space E_i;*

(ii) *A_i is continuous with non-empty compact convex values;*

(iii) *the mapping $A\colon X = \Pi_{i\in I} X_i \to 2^X$ defined by $A(x) = \Pi_{i\in I} A_i(x)$ for each $x \in X$ is Ψ-condensing, where $\Psi : 2^{\Pi_{j\in I} E_j} \to C$ is a measure of non-compactness;*

(iv) *the set $U_i := \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is either open or closed in X;*

(v) *the mapping $A_i \cap P_i$ is lower semicontinuous on U_i such that for each $x \in U_i$, $A_i(x) \cap P_i(x)$ is either convex finite dimensional (not necessarily closed), or convex closed, or has an interior point (not necessarily closed).*

Then there exists $x^* \in X$ such that for each $i \in I$, either $\pi_i(x^*) \in A_i(x^*) \cap P_i(x^*)$ or $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

Remark 3.1. We thank the anonymous referee to bring our attention of the statement of the condition (v) in Theorem 3.5.

As an application of Theorem 3.5, we have the following

Corollary 3.6. *Let I be a countable set. For each $i \in I$, let X_i be a non-empty compact convex subset of a finite dimensional space E_i and $P_i : X = \Pi_{j\in I} X_j \to 2^{X_i}$ be lower semicontinuous on the set $U_i = \{x \in X : P_i(x) \neq \emptyset\}$ such that for each $x \in U_i$, $P_i(x)$ is convex. If for each $i \in I$, U_i is either open or closed in X, then there exists $x^* \in X$ such that for each $i \in I$ either $\pi_i(x^*) \in P_i(x^*)$ or $P_i(x^*) = \emptyset$.*

Proof. For each $i \in I$, let $A_i : X \to 2^{X_i}$ be defined by $A_i(x) = X_i$ for each $x \in X$. Then A_i is continuous with compact convex values and A_i is also Ψ-condensing, where $\Psi : 2^{\Pi_{i\in I} E_i} \to \mathbb{R}$ is the ball-measure (or set-measure) of non-compactness on $\Pi_{i\in I} E_i$ (which is a metric space as I is countable). Therefore by Theorem 3.5, there exists $x^* \in X$ such that for each $i \in I$ either $\pi_i(x^*) \in P_i(x^*)$ or $P_i(x^*) = \emptyset$. $\square$

Corollary 3.6 generalizes Theorem 1 of Barbolla [2, p. 206] which in turn improves the fixed point theorem of Gale and Mas-Colell [10, p. 10] and Florenzano [8] in the following ways: (1) the index set I is countable instead of finite and (2) for each $i \in I$, U_i is either open or closed instead of U_i being open for all $i \in I$ or U_i being closed for all $i \in I$. We remark that our arguments in proving Theorem 3.4 and Theorem 3.5 are different from that of Barbolla [2].

For the existence of equilibrium point of abstract economies in which the constraint correspondences are not condensing in topological vector spaces, we refer [22–26] and the references wherein to the reader.

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