On locally C^*-algebras

By B. D. MALVIYA (Denton, Texas)

1. Introduction

As is well known the concept of Banach algebras have been generalized to locally m-convex algebras by MICHAEL [5]. He obtained many results on locally m-convex algebras with and without involution. Some results concerning spectrum, topological divisors of zero, and the representations of locally m-convex *-algebra have been obtained in [2], [3], [4]. INOUE [1] has recently introduced the notion of locally C^*-algebras. The main objective of the present paper is to obtain further results on locally C^*-algebras by using some of the techniques developed by INOUE [1]. In particular we consider the representation of a locally C^*-algebra on a Hilbert space. We also prove that under certain conditions a locally C^*-algebra admits a complete set of irreducible representations. An inequality involving the positive functionals on a locally C^*-algebra and spectral radius is also obtained.

2. Preliminaries

A topological algebra is an algebra and a topological linear space such that the ring multiplication is continuous. A locally convex algebra is a topological algebra and a locally convex linear space. A topological algebra A with usual involution * such that $x \to x^* (x \in A)$ is continuous, is called topological *-algebra. A locally m-convex algebra (respectively *-algebra) A is a Hausdorff topological vector space which is an algebra (respectively a *-algebra for which $x \to x^*$ is continuous) having a fundamental family (U) of circled convex neighbourhoods of $0 \in A$ such that $UU \subset U$. It is clear that every locally m-convex algebra is a locally convex algebra with jointly continuous ring multiplication. The topology in a locally m-convex algebra can be generated by a collection $\{p_j : j \in J\}$ of semi-norms satisfying $p_j(xy) \leqq p_j(x) p_j(y)$ for all $x, y \in A$ and all $j \in J$. By an m-base in a locally m-convex algebra A, we understand the collection $\{U\}$ of subsets of A such that U is convex, symmetric and scalar multiples of $\{U\}$ form a base for the neighbourhood of the origin.

Following [1] we define a locally C^*-algebra as follows.

Definition 1.1. A *-algebra A is called a locally C^*-algebra if there exists a family of semi-norms $\{p_j : j \in J\}$ defined on A such that

1. $\{p_j\}$ defines a complete Hausdorff locally convex topology on A.
2. $p_j(xy) \leqq p_j(x)p_j(y)$ for all $x, y \in A$ and all $j \in J$.
3. $p_j(x^*) = p_j(x)$ for all $x \in A$ and all $j \in J$.
4. $p_j(x^*x) = p_j(x)^2$ for all $x \in A$ and all $j \in J$.

It may be mentioned that a closed *-subalgebra of a locally C^*-algebra is also a locally C^*-algebra. Our locally C^*-algebras are always complete. Let A be a locally C^*-algebra. Define $N_j = \{x \in A : p_j(x) = 0\}$. Obviously N_j is a closed ideal in A. Let $A_j = A/N_j$ be given a norm V_j defined by $V_j(x+N_j) = p_j(x)$ for $x \in A$. Then A_j is a normed *-algebra with the norm as defined above; and furthermore $\|x_j^* x_j\| = \|x_j\|^2$, where x is any pre-image of x_j under the natural homomorphism π_j. The completion $\bar{A}_j$ of A_j is a C^*-algebra. We shall follow the terminology and notations used by Inoue [1] as closely as possible. For general terms not defined in this paper a reference is made to Rickart [7].

Let A be an algebra. An element $x \in A$ is said to be left (respectively right) quasi-regular if there exists an element $y \in A$ such that $x+y-yx = 0$ (respectively $x+y-xy = 0$). An element which is both left and right quasi-regular is simply called quasi-regular. An element which is not quasi-regular is called quasi-singular. The spectrum of an element $x \in A$, written $\mathrm{Sp}_A(x)$, is above $\{0\}$ the set $\{\lambda : \lambda^{-1}x$ is quasi-singular in $A\}$. The spectral radius of an element $x \in A$, written $v_A(x)$, is defined as $v_A(x) = \max \{|\lambda| : \lambda \in \mathrm{Sp}_A(x)\}$. The spectral radius of an element x of a complete locally m-convex algebra A is given by $v_A(x) = \sup_j \lim_{n\to\infty} (p_j(x^n))^{1/n}$ [5, Corollary 5.3, p. 22].

3. Spectrum

This section mainly concerns with theorems related to spectrum of an element in a locally C^*-algebra. The set of all complex homomorphism of a commutative algebra A is denoted by Φ_A. For a commutative locally C^*-algebra A, the map $x \to \hat{x}$ defined by $\hat{x}(\varphi) = \varphi(x)$, $(x \in A, \varphi \in \Phi_A)$, is called the Gelfand map. The set of hermitian elements of a *-algebra A is denoted by H_A. With the aid of [1, Theorem 3.1, p. 212 and Proposition 3.1, p. 211], it can be proved that the spectrum of an element x of a complete, commutative locally m-convex algebra A is given by the set $\{\hat{x}(\varphi) : \varphi \in \Phi_A\}$. Observe that an element $x \in A$ is quasi-regular if and only if $\hat{x}(\varphi) \neq 1$.

Definition 3.1. A *-algebra is called symmetric if x^*x is quasi-regular for every $x \in A$.

Theorem 3.2. *Let A be a locally C^*-algebra with the family $\{p_j\}$ $j \in J$ of semi-norms and B a symmetric sublocally C^*-algebra of A. If $x \in B$ is quasi-regular in A, then x is also quasi-regular in B.*

Proof. Since $h = x^* 0 x (= x^* + x - x^* x)$ is quasi-regular in A, therefore $1 \notin \mathrm{Sp}_A(h)$. Now by [1, Corollary 2.2, p. 200], $\mathrm{Sp}_A(h)$ is real. Again using [1, Corollary 2.3, p. 200], it follows that $\mathrm{Sp}_A(h) = \mathrm{Sp}_B(h)$. Hence $1 \notin \mathrm{Sp}_B(h)$ and therefore h is quasi-regular in B. Similar statement holds for $x^* 0 x$. Hence the theorem is proved.

Theorem 3.3. *Let A be a commutative and symmetric locally C^*-algebra without an identity element and A_1, the *-algebra obtained by adjunction of an identity element to A. Then A_1 is symmetric.*

PROOF. The fact that A_1 is a locally C^*-algebra follows from [1, Theorem 2.3, p. 201]. For an arbitrary element $y = \xi + x \in A_1$, put $h = \bar{\xi}x^* + \xi x + x^* x$. Then $h \in H_A$. Let C be a maximal *-subalgebra of A containing h. It is easy to see that C is a closed *-subalgebra of A and therefore a symmetric sublocally C^*-algebra of A. Let φ be an arbitrary element Φ_{A_1}. Then by [1, Corollary 2.3, p. 200] applied to C, we have $\widehat{y * y}(\varphi) = |\xi|^2 + \hat{h}(\varphi) \neq 1$. Hence the theorem is proved.

4. Representations

As mentioned in the introduction we shall deal with the positive functionals and *-representations on a locally C^*-algebra. We shall use the same notations as employed by Inoue [1, pp. 211—213].

Let A be a locally C^*-algebra with a family $\{p_j\}$, $j \in J$ of semi-norms. Define $U_j = \{x \in A : p_j(x) \leqq 1\}$. A^* and $A^*(j)$ will denote respectively the conjugate space of A and all linear functionals on A that are bounded on U_j.

Let A be a *-algebra and f a positive functional, that is $f(x^*x) \geqq 0$ for each $x \in A$. Define $(x, y) = f(y^*x)$. The following is well know for any $x, y \in A$.

(i) $$f(y^*x) = \overline{f(x^*y)}.$$

(ii) $$|f(x^*y)|^2 \leqq f(x^*x)f(y^*y).$$

In presence of an identity element, putting $y = e$ in (i) and (ii), we get

(iii) $$f(x^*) = \overline{f(x)}.$$

(iv) $$|f(x)|^2 \leqq f(e)f(x^*x).$$

A positive functional f on A is said to be admissible if

$$\sup_{x \in A} \frac{f(x^* a^* ax)}{f(x^* x)} < \infty,$$

for each x, $a \in A$ and $f(x^*x) \neq 0$.

Theorem 4.1. *Let A be a locally C^*-algebra with an identity element e and $f \in A^*(j)$. Then*

$$|f(x)|^2 \leqq \big(f(e)\big)^2 \nu(x^*x)$$

for each $x \in A$.

PROOF. From the proof of Lemma 4.1.4 in [7, p. 183] it follows that there exists an element $y \in A$ such that $y^2 = e - x$ for $x \in A$, provided $p_j(x^n)^{1/n} < 1$ $\big($or $\nu(x) < 1\big)$; see [5, Corollary 5.3, p. 22]. It also follows from the lemma cited above that if x

is hermitian, then y is also hermitian and $y^2 = y^*y = e-x$. Let $\varepsilon>0$ be given. Then $v\left(\frac{x}{v(x)+\varepsilon}\right) < 1$ and we have

$$y^2 = e - \frac{x}{v(x)+\varepsilon}.$$

If x is hermitian, then

$$0 \leqq f(y^*y) \leqq f(e) - \frac{f(x)}{v(x)+\varepsilon}.$$

Since ε is arbitrary, we get

$$f(x) \leqq f(e)v(x).$$

Replacing x by x^*x, we get

(1) $$f(x^*x) \leqq f(e)\,v(x^*x).$$

By [1, Theorem 3.2, p. 213], we have

(2) $$|f(x)|^2 \leqq \|f_j\| f(x^*x)$$

for each $x \in A$ and $j \in J(f)$, where $J(f) = \{j \in J : f \text{ bounded on } U_j\}$.

From (1) and (2),

$$|f(x)|^2 \leqq f(e)\,\|f_j\|\,v(x^*x).$$

Using [1, Corollary 3.2, p. 216], we get

$$|f(x)|^2 \leqq (f(e))^2\,v(x^*x),$$

which is the desired inequality.

Let f be a positive functional on a^*-algebra A. Define $N_f = \{x \in A : f(x^*x) = 0\}$. Obviously N_f is a left ideal in A. For elements x_f and y_f in the difference space A/N_f; define $f(y^*x) = (x_f, y_f)$. It is easily verified that A/N_f with this inner product is a pre-Hilbert space. The completion of A/N_f will be denoted by H_f.

In the theorems to follow a reference to Hilbert space means the Hilbert space H_f as constructed above.

Definition 4.2. A representation $a \to T_a$ of a *-algebra A on a Hilbert space is called a *-representation if $T_{a^*} = T_a^*$, where T_a^* is the adjoint of T_a; $a \in A$.

Definition 4.3. A representation $a \to T_a$ of an algebra A on a Hilbert space H is called topologically cyclic if there exists a vector $f \in H$ such that the linear subspace $\{T_a f : a \in A\}$ is dense in H.

Theorem 4.4. *Let A be a locally C^*-algebra with an identity element e. Suppose that A has bounded spectrum. Then there exists a *-representation $a \to T_a$ of A on a Hilbert space which is topologically cyclic with a cyclic vector ζ such that $f(a) = (T_a\zeta, \zeta)$, where $f \in A^*(j)$ and is a positive functional.*

Proof. The construction of Hilbert space is obvious as indicated before. Let us write the inner product as $f(y^*x) = (x_f, y_f)$. Define an operator T_x on A/N_f by

$T_x(y+N_f)=xy+N_f$. The mapping $x\to T_x$ is obviously a representation. Moreover, T_x, $x\in A$, is bounded, since for $f_0\in A/N_f$, $f_0=y+N_f$ we have

$$(T_x f_0, T_x f_0) = (xy+N_f, xy+N_f) = \big((xy)_f, (xy)_f\big) = f\big((xy)^* xy\big) \leqq$$
$$\leqq \big(f(e)\big)^2 \nu\big((xy^* xy)^2\big),$$

by Theorem 4.1 above. The extension of T_x to H_f is bounded and unique. The following identity shows that $x\to T_x$ is a *-representation on A/N_f and hence on H_f. For $x_f, y_f\in A/N_f$, we have

$$(T_a x_f, y_f)=f(y^* ax)=f\big((a^* y)^* x\big)=(x_f, T_{a^*} y_f).$$

Now by [1, Theorem 3.2 (2), p. 213], f is admissible; see [7, p. 213]. Hence our theorem follows by an application of [7, Theorem 4.5.4 (iii), p. 215], noting that f is a positive hermitian functional.

We shall remove the identity restriction from Theorem 4.4 above. The details of the proof are the same as in Theorem 4.4, so we only give a skeleton of the proof.

Theorem 4.5. *Let A be a locally C^*-algebra without an identity element and with bounded spectrum. Let $f\in A^*(j)$ be a positive functional. Then there exists a *-representation $x\to T_x$ of A on a Hilbert space which is topologically cyclic with a cyclic vector ζ_2 such that $f(x)=(T_x\zeta_2, \zeta_2)$.*

Proof. Let f be a positive functional on A. Let A_1 be the locally C^*-algebra obtained by adjunction of an identity element; see [1, Theorem 2.3, p. 200]. By [1, Theorem 3.2, p. 213], f can be extended uniquely to $\tilde{f}$. Now as in the proof of Theorem 4.4 above, $f(x)=(T_x\zeta, \zeta)$, where $\zeta\in H_f$ is a cyclic vector. Here $x\to T_x$ is a *-representation corresponding to A_1. We are now looking for a Hilbert space and a cyclic vector when $\tilde{f}$ is restricted to A. Define $H_1=\{\xi\in H: T_x\xi=\xi \text{ for all } x\in A\}$. Write $\zeta=\zeta_1+\zeta_2$, where $\zeta_1\in H_1$ and $\zeta_2\in H_1^{\perp}$ (the orthogonal complement of H_1). Observing the fact that the under the *-representation $x\to T_x$, H is invariant, we have $f(x)=(T_x\zeta_2, \zeta_2)$. Define $H_0=\overline{\{T_x\zeta_2: x\in A\}}$. Obviously H_0 is invariant. By [6, Lemma 4.4.1, p. 206], we have a cyclic vector $\zeta_2\in H_0$. This proves our assertion completely.

Definition 4.6. The set of representations is said to be complete if for every element $x_0\neq 0$ of the algebra A, there exists an irreducible representation $x\to T_x$ such that $T_{x_0}\neq 0$.

We now obtain a theorem corresponding to Theorem 3 of [5, p. 267].

Theorem 4.7. *Let A be a locally C^*-algebra with an m-base. A has a complete set of irreducible representations provided there exist positive functionals f such that $M=N_f$, where M's are maximal modular ideals in A.*

Proof. As in Theorem 4.4, $x\to T_x$ is a *-representation of A on A/N_f. By [4, Theorem 8.4, p. 36], A is semi-simple. Let $0\neq a\in A$. Let us choose a positive functional f such that $a\notin M$. Define $T_x(y+N_f) = xy+N_f$. Obviously $x\to T_x$ is irreducible. But the kernel of the homomorphism $x\to T_x$ is contained in $N_f=M$, therefore $T_a\neq 0$. Hence the theorem is proved.

Lemma 4.8. *Let A be a locally C^*-algebra with an identity element e. Suppose that A has bounded spectrum. Then $\|T_x\|^2 \leqq \nu(x^* x)$ for each $x \in A$, where $x \to T_x$ is a representation of A.*

PROOF. As in the proof of Theorem 4.4, we have $f(x) = (T_x \zeta, \zeta)$, where $f \in A^*(J)$. Now

$$f(x^* x = (T_{x^* x} \zeta, \zeta) \leqq f(e)\, \nu(x^* x),$$

from inequality (1) in the proof of Theorem 4.1. This implies

$$\|T_x \zeta\|^2 \leqq \|\zeta\|^2 \nu(x^* x)$$

or

$$\|T_x\|^2 \leqq \nu(x^* x).$$

Now we state the following theorem about the direct sum of the *-representations. The construction of proof is similar to that in [7, pp. 197—198] and makes use of Lemma 4.8.

Theorem 4.9. *Let A be a locally C^*-algebra with an identity element and bounded spectrum. Then the direct sum of the *-representations is defined**

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