On the bases for laws of finite groups of small orders

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§ 1. Preliminaires

The finite basis problem, 'whether all the laws of a given group are derivable from a finite set', is found to hold in affirmative for considerably many cases. One of them is' every finite group has a finite basis for its laws (cf. [1], 52. 12).

In this note we work out the bases for laws of finite groups of small orders, namely $\leqq 15$. However, one can obtain even beyond 15 in the light of this note.*)

We shall denote by $\mathrm{Var}\,(w_1, \dots, w_r)$, the variety generated by the laws $w_1 = 1, \dots, w_r = 1$. Rest of the notations are adopted from [1].

The key result of this section is the following:

Theorem 1.1. (12. 12. [1]). *Every word w (the left hand side of the law $w=1$) is equivalent to a pair of words, one of the form x^m, $m \geqq 0$ and the other a commutative word.*

Corollary 1.2. $\mathfrak{A}_n$, an abelian variety of exponent n, has the basis for its laws x^n, $[x, y]$.

The laws of all cyclic groups C_n $(1 \leqq n \leqq 15)$, abelian groups of order 4 and 9, abelian groups of order 8 ($C_4 \times C_2$ and $C_2 \times C_2 \times C_2$) and abelian group of order 12 ($C_2 \times C_2 \times C_3$) are easily obtainable by corollary 1.2. Hence we only tabulate the non-abelian groups, namely, Q_8, D_4, D_3, D_5, D_6, D_7, A_4 and $M = \mathrm{gp}\,\{x, y \mid x^3 = 1, y^4 = 1, x^y = x^{-1}, x^{y^{-1}} = x^{-1}\}$.

§ 2. Variety generated by the given group

In this section we study what varieties are generated by the groups mentioned in Table 1.3.

Lemma 2.1. (54. 23, [1]). $\mathrm{Var}\, Q_8 = \mathrm{Var}\, D_4$.

*) DR. SHEILA MACDONALD has informed me (with the comments on my manuscript) that her student MR. RICHARD LEVINGSTON has been successful in enhancing my work till the order <32 in the light of my note and [3] p. 134. Any way I record Mr. Levingston's gratitude for drawing my attention to his work through his supervisor.

1.3. *Table Non-abelian groups*

Order n	Nature	Presentations
1. $n=8$	Q_8	$\mathrm{gp}\,\{x, y \mid x^4=1,\ y^2=x^2,\ x^y=x^{-1}\}$
2. $n=2m,\ 3 \leqq m \leqq 7$	D_m	$\mathrm{gp}\,\{x, y \mid x^m=1,\ y^2=1,\ x^y=x^{-1}\}$
3. $n=12$	A_4	$\mathrm{gp}\,\{x, y \mid x^2=y^3=(xy)^3=1\}$
4. $n=12$	M	$\mathrm{gp}\,\{x, y \mid x^3=1,\ y^4=1,\ x^y=x^{-1},\ x^{y^{-1}}=x^{-1}\}$

Theorem 2.2. Var $D_4=\mathfrak{A}_2^2 \wedge \mathfrak{N}_2$.

PROOF. D_4 is clearly metabelian group of exponent 4, i.e., Var $D_4 \subseteq \mathfrak{A}_2^2$. Further we note that it is nilpotent of class 2, i.e., $[x_1, x_2, x_3]$ is a law in D_4, hence Var $D_4 \subseteq \mathfrak{N}_2$. Concludingly Var $D_4 \subseteq \mathfrak{A}_2^2 \wedge \mathfrak{N}_2$. On the other hand if we consider a group $G \in \mathfrak{A}_2^2 \wedge \mathfrak{N}_2$ that generates the variety $\mathfrak{A}_2^2 \wedge \mathfrak{N}_2$, then $|G|$*$)=8$, hence $G \in$ Var D_4, that is, $\mathfrak{A}_2^2 \wedge \mathfrak{N}_2 \subseteq$ Var D_4. Thus we have, Var $D_4=\mathfrak{A}_2^2 \wedge \mathfrak{N}_2$.

Remark: Var $D_4 \neq \mathfrak{A}_2^2$ as $\mathfrak{A}_2^2$ is not generated by any one of its finitely generated free groups in view of 16.36 of [1].

Theorem 2.3. Var $D_p=\mathfrak{A}_p\mathfrak{A}_2$ *where p is an odd prime.*

PROOF. By presentation of D_p; $D_p \in \mathfrak{A}_p \cdot \mathfrak{A}_2$, hence Var $D_p \subseteq \mathfrak{A}_p\mathfrak{A}_2$. Conversely consider any group G of order $2p$ in $\mathfrak{A}_p\mathfrak{A}_2$. Obviously this group G is non-abelian of order $2p$ hence isomorphic to D_p. Therefore $\mathfrak{A}_p\mathfrak{A}_2 \subseteq$ Var D_p. Hence the theorem.

Corollary 2.4. Var $D_3=\mathfrak{A}_3 \cdot \mathfrak{A}_2=$ Var D_6.

PROOF. Allowing $p=3$ in theorem 2.3, we have Var $D_3=\mathfrak{A}_3\mathfrak{A}_2$. But since $D_6=D_3 \times C_2$ (cf. Problem 5.38 (iii) [2]) and C_2 is the subgroup of D_3 so Var $D_6=$ Var D_3.

Theorem 2.5. Var $A_4=\mathfrak{A}_2\mathfrak{A}_3$.

PROOF. Since $A_4=(C_2 \times C_2)$ — by — C_3, so Var $A_4 \subseteq \mathfrak{A}_2 \cdot \mathfrak{A}_3$. In view of 24.64 and 15.61 of [1] $\mathfrak{A}_2\mathfrak{A}_3$ is locally finite. So 51.41, [1]; guarantees that $\mathfrak{A}_2\mathfrak{A}_3$ is generated by its critical groups. Clearly any critical group of $\mathfrak{A}_2\mathfrak{A}_3$ is of order 12*). Now to choose G as D_6 is out of question because D_6 will not generate $\mathfrak{A}_2 \cdot \mathfrak{A}_3$. Also since M (cf. Table 1.3) is of exponent 12 so $G \not\cong M$. Thus $G \in$ Var A_4 or Var $G=\mathfrak{A}_2\mathfrak{A}_3 \subseteq$ Var A_4. This completes the proof.

Theorem 2.6. Var $M=\mathfrak{A}_3 \cdot \mathfrak{A}_2 \vee \mathfrak{A}_4$.

*) $|G|=8$ is trivial. Because G is evidently a 2-group and $|G|>2^2$. Thus either $|G|>2^3$ or $|G| \leqq 2^3$. For the first case $|G|=2^{3+\alpha}$, $\alpha \geqq 1$, without loss of generality let $\alpha=1$ so that $|G|=2^4$, hence G has a monolith factor D_4 so in view of 53.72 of [1] it has exponent 8 i.e., $G \notin \mathfrak{A}_2^2$, a contradiction. In second case $|G|$ is exactly 8 as G is non-abelian and $|G|=2^2$ is impossible.

*) Clearly $|G|=2^\alpha \cdot 3^\beta$. As G is of exponent 6 the maximum value of β is 1. Now α can achieve value >1, as $\alpha=1$ gives $G=D_3 \cong S_3$ which means $G \in \mathfrak{A}_2\mathfrak{A}_3$. Further $\alpha>2$ asserts G is of exponent 12 in view of 53.72 of [1]. Hence only possible value of α is 2 and $|G|=12$.

PROOF. Clearly $\mathfrak{A}_3\mathfrak{A}_2 \vee \mathfrak{A}_4 \subseteqq \mathrm{Var}\, M$. On the other hand M is a split extension of C_3 — by — C_4 so in view of 24.62, [1], Var M is locally finite, as it is metabelian of exponent 12. Hence Var M is generated by its critical group G. Obviously G may have order 12, 6 and 4. But in view of lemma 51.37 of [1] M is not critical so G may have order 4 and 6 only. M, a fortiori, is the subdirect product of C_4 and S_3. So critical groups are S_3 and C_4. Hence Var $M \subseteqq \mathfrak{A}_3\mathfrak{A}_2 \vee \mathfrak{A}_4$. This completes the proof of the theorem.

§ 3. The laws characterizing a group

In order to study the laws characterizing a group A we study the laws in Var A. In § 2 we have studied $\mathfrak{A}_2^2 \wedge \mathfrak{N}_2$, $\mathfrak{A}_p\mathfrak{A}_2$, $\mathfrak{A}_p\mathfrak{A}_q$ (for example $\mathfrak{A}_2\mathfrak{A}_3$) where $q \neq 2$; p, q are distinct primes and $\mathfrak{A}_3\mathfrak{A}_2 \vee \mathfrak{A}_4$. We study the laws of these varieties.

Theorem 3.1. $\mathfrak{A}_2^2 \wedge \mathfrak{N}_2 = \mathrm{Var}\,\{x^4, [x^2, y]\}$.

PROOF: Clearly var $\{x^4, [x^2, y]\} \subseteqq \mathfrak{A}_2^2 \wedge \mathfrak{N}_2$. But $\mathfrak{A}_2^2 \wedge \mathfrak{N}_2 = \mathrm{var}\,\{(x^2y^2)^2, [x, y, z]\} = \mathrm{var}\,\{x^4, [x, y]^2, [x^2, y^2], [x, y, z]\}$. So to establish the reverse inclusion it suffices to show that $[x, y, z] \Rightarrow [x^2, y]$. Clearly $[x^2, y] = x^{-1} \cdot [x, y] \cdot x \cdot [x, y]$ by 33.34 of [1]. Also $x^{-1} \cdot [x, y] \cdot x \cdot [x, y] = x^{-1} \cdot [x, y]^{-1} \cdot x \cdot [x, y]$; as commutators have order 2 in $\mathfrak{A}_2^2 \wedge \mathfrak{N}_2 = \big[x, [x, y]\big]$. So $\big[x, [x, y]\big] = [x^2, y]$. But $[x, y, z] \Rightarrow \big[x, [x, y]\big]$. Hence $[x, y, z] \Rightarrow [x^2, y]$, concluding $\mathfrak{A}_2^2 \wedge \mathfrak{N}_2 \subseteqq$ var $\{x^4, [x^2, y]\}$.

Theorem 3.2. $\mathfrak{A}_p \cdot \mathfrak{A}_2 = \mathrm{var}\,\{x^{2p}, [x^2, y^2]\}$.

PROOF. The sets for laws of $\mathfrak{A}_p$ and $\mathfrak{A}_2$ are $U = \{x^p, [x, y]\}$ and $V = \{x^2, [x, y]\}$ respectively. But as $x^2 \Rightarrow [x, y]$ so $V = \{x^2\}$. In view of 21.12 of [1] the basis for laws of $\mathfrak{A}_p\mathfrak{A}_2$ is $U(V) = \{x^{2p}, [x^2, y^2]\}$.

Theorem 3.3. $\mathfrak{A}_p \cdot \mathfrak{A}_q = \mathrm{Var}\,\{x^{pq}, [x, y]^p, [x^q, y^q]\}$ *where p, q are distinct primes, $q \neq 2$.*

PROOF. The sets for laws of $\mathfrak{A}_p$ and $\mathfrak{A}_q$ are $U = \{x^p, [x, y]\}$ and $V = \{x^q, [x, y]\}$ respectively. So laws of $\mathfrak{A}_p\mathfrak{A}_q$ are given by the set

$$U(V) = \{x^{pq}, [x^q, y^q], [x, y]^p, \big[[x, y], [z, w]\big]\}.$$

But $\big[[x, y], [z, w]\big]$ is implied by $[x^2, y^2]$ indicating that q would be 2, a contradiction. Thus $U(V) = \{x^{pq}, [x^q, y^q], [x, y]^p\}$ is the required basis for laws, proving the theorem.

Theorem 3.4. $\mathfrak{A}_3 \cdot \mathfrak{A}_2 \vee \mathfrak{A}_4 = \mathrm{Var}\,\{x^{12}, [x, y]^3, [x^2, y^2], [x^6, y]\}$.

PROOF. In view of 15.83 it is immediate by inspection.

§ 4. Bases for the laws

Groups of order 1 are all isomorphic to the trivial group for which the basis for laws is x. For all abelian groups (cyclic or non-cyclic) of order n the basis for laws is $\{x^n, [x, y]\}$. We tabulate here the laws of different non-abelian groups of order $\leqq 15$.

S. No	Group	Order	Basis for laws
1	D_3	6	$x^6, \quad [x^2, y^2]$
2	D_6	12	$x^6, \quad [x^2, y^2]$
3	D_4, Q_8	8	$x^4, \quad [x^2, y]$
4	D_5	10	$x^{10}, \quad [x^2, y^2]$
5	D_7	14	$x^{14}, \quad [x^2, y^2]$
6	A_4	12	$x^6, \quad [x^3, y^3], \quad [x, y]^2$
7	M	12	$x^{12}, \quad [x, y]^3, \quad [x^2, y^2], \quad [x^6, y]$

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References

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