Reduction of all three-term syzygies

By ELFRIEDE WENZELBURGER and JENG-ENG LIN (Austin, Tex.)

The purpose of this paper is to present a different approach to a problem which was discussed by M. HOSSZÚ [2]. We consider a single-valued binary operation defined on a set S, which we denote by xy, where x, y and xy belong to S.

The associative law states that

$$1. \quad x(yz) = (xy)z.$$

If we interchange the order of the neighbouring factors in the multiplications appearing in 1., we get 24 equations which we will call three-term syzygies. We are going to adopt two criteria according to which any two of the below listed equations will be considered as equivalent. In this way we will find a smallest number of different syzygies. This number cannot be further reduced by our criteria.

We will prove the independence of those syzygies, i.e. we will prove that no one of them implies any other. We will also discuss their relation to mediality.

First we will list the 24 syzygies which represent all possible variations of the associative law:

	13. $x(yz) = x(yz)$
2. $(xy)z = x(zy)$	14. $x(yz) = x(zy)$
3. $(xy)z = y(xz)$	15. $x(yz) = y(xz)$
4. $(xy)z = y(zx)$	16. $x(yz) = y(zx)$
5. $(xy)z = z(xy)$	17. $x(yz) = z(xy)$
6. $(xy)z = z(yx)$	18. $x(yz) = z(yx)$
7. $(xy)z = (yz)x$	19. $x(yz) = (yz)x$
8. $(xy)z = (zy)x$	20. $x(yz) = (zy)x$
9. $(xy)z = (xz)y$	21. $x(yz) = (xz)y$
10. $(xy)z = (zx)y$	22. $x(yz) = (zx)y$
11. $(xy)z = (xy)z$	23. $x(yz) = (xy)z$
12. $(xy)z = (yx)z$	24. $x(yz) = (yx)z$

We can eliminate immediately 11 and 13 as trivial from our list and also 23 since it coincides with 1.

We can reduce the number of 21 syzygies we have now, to 14 by simply observing that some of them appear actually repeated. This is the case for 22 and 4, 19 and 5, 6 and 20, 16 and 17, 10 and 7, 2 and 21, and 24 and 3. We choose 1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 14, 15, 16 and 18 as representatives.

To reduce the number of 14 syzygies further we introduce the notation $t^*s=st$. For example, we consider 2. $(xy)z=x(zy)$. It transforms into $(y^*z)^*x=z^*(y^*x)$ which is 3. So we state that 2 is the dual of 3. Similarly we can relate 7 and 16, 18 and 8, 9 and 15, and 12 and 14. We observe that 1, 4, 5 and 6 are self-dual.

As our final list of nine three-term syzygies which are different according to our criteria we choose 1, 2, 4, 5, 6, 7, 8, 9 and 12. We denote their duals as 1^*, 2^* etc.

Our next result shows the independence of the nine syzygies.

We are going to exhibit some examples in which some of them hold and some of the remaining ones are not satisfied.

Example I: Let $S=S_3$, the symmetric group of order 3.

Example II: Let $S=\{a, b, c\}$, the binary operation is defined by $aa=ab=a$, $ba=bb=b$, $xy=c$ for all other cases.

Example III: Let $S=\{a, b, c\}$, the binary operation is defined by $aa=ab=ac=a$, $ba=ca=c$, $xy=b$ otherwise.

Example IV: Let $S=\{0, a, b, c, d, e\}$, let $be=dc=a$, $dd=b$, $ed=c$, $xy=0$ otherwise. (This example is due to Reverdy Wright.)

Example V: Let $S=\{a, b, c, d, f, g, h, i, j, k, m, n, p, q, 0\}$, let $ab=e$, $ac=j$, $ba=m$, $bd=q$, $cd=f$, $cm=k$, $cq=p$, $dc=n$, $de=g$, $dj=i$, $fa=i$, $jb=k$, $nb=p$, $qa=q$, $ap=bi=ef=gc=kd=mn=h$, and $xy=0$ otherwise.

Example VI: Let $S=\{a, b, c\}$, let $aa=bb=cc=a$, $xy=c$ otherwise.

Example VII: Let $S=\{a, b, c, d\}$, let $bb=bc=bd=cb=cc=cd=dc=c$, $xy=a$ otherwise.

Example VIII: Let $S=\{a, b, c, d, e, f, 0\}$, let $ab=d$, $dc=f$, $ba=e$, $ce=f$ and $xy=0$ otherwise.

Example IX: Let $S=\{a, b, c\}$, let $ac=b$, $xy=c$ otherwise.

Example X: Let $S=\{a, b, c, d, f, g, h, i, j, k, m, n, p, q, 0\}$; let $ab=e, ac=n$, $bf=g$, $bj=m$, $bn=q$, $cb=k$, $cd=f$, $da=j$, $db=p$, $fa=jc=nd=i$, $ef=ga=ib=kj=$ $=mc=pn=qd=h$, and $xy=0$ otherwise.

Example XI: Let $S=\{a, b, c\}$, $ab=b$, $ba=a$, $xy=c$ otherwise.

Example XII: Let $S=Z$, the set of integers, define $xy = x-y+xy$ for every x, y in Z.

We will make use of these examples in the proof of our first proposition.

Proposition 1: *Among* 1, 2, 4, 5, 6, 7, 8, 9 *and* 12, *no one of them implies any other.*

PROOF. Example I shows that 1 does not imply any other syzygy among the nine. Example (Ex.) II and III show that 2 does not imply any other, Ex. IV and V show it for 4, Ex. VI and VII show it for 5, Ex. VI and VIII show it for 6, Ex. IX X show it for 7, Ex. IX and XI show it for 8, Ex. XII shows it for 9 and Ex. VI and IX show it for 12.

After having discussed the independence of the nine syzygies we will prove a result related to sets with medial (bisymmetric) binary operations.

Definition 1: A binary operation is medial if and only if $(xy)(uv)=(xu)(yv)$.

Proposition 2: *In a set with a binary operation, each of* 8 *and* 8^* *implies mediality and none of the remaining syzygies does. Commutativity doas not imply mediality*

PROOF. Suppose 8 holds, then $(xy)z=(zy)x$. Now

$$(xy)(uv) = v(u(xy)) = v(y(xu)) = (xu)(yv).$$

Suppose now 8^* holds, then $x(yz)=z(yx)$. We have $(xy)(uv)=v(u(xy))=v(y(xu))=$ $=(xu)(yv)$. This proves the first assertion.

Example I shows that 1 does not imply mediality, Ex. III shows it for 2, Ex. V for 4, Ex. VI for 5, 6 and 12, Ex. X shows it for 7 and Ex. XII for 9.

Finally Example VI proves that commutativity does not imply mediality.

This completes the proof.

References

[1] T. FARAGÓ, Contribution to the definition of group, *Publ. Math. (Debrecen)* **3** (1953), 133—137.
[2] M. HOSSZÚ, Some functional equations related with the associative law. *Publ. Math. (Debrecen)*, **3** (1954), 205—214.

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