On the absolute Riesz summability of Fourier series

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The object of this paper is to generalize a recent Theorem of PREM CHANDRA on absolute Riesz summability of Fourier series. It has also been shown that one of his conditions is redundant.

1. Let $\sum_{n=0}^{\infty} a_n$ pe a given infinite series with the sequence of partial sums $\{S_n\}$ and let $\{\lambda_n\}$ be an increasing sequence of positive numbers tending to infinity with n and

$$\lambda_n = \mu_0 + \mu_1 + \mu_2 + \dots + \mu_n,$$

$$t_n = \frac{1}{\lambda_n} \sum_{v=0}^{n} \mu_v S_v.$$

A series $\sum_n a_n$ is said to be *summable* $|R, \lambda_n, 1|$ if*) $t_n \in \mathrm{BV}$ and write $\sum_{n=0}^{\infty} a_n \in |R, \lambda_n, 1|$.

Let $f(t)$ be L-integrable in $(-\pi, \pi)$ with period 2π. Without any loss of generality the constant term of the Fourier series of $f(t)$ can be taken to be zero, so that

$$\int_{-\pi}^{\pi} f(t)\, dt = 0,$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

We shall use the following notations:

$$\Phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\},$$

$$\Lambda(t) = \frac{1}{t} \int_0^t u\, d\Phi(u),$$

$$K(n, t) = \sum_{v=0}^{n} \frac{\lambda_v}{v+1} \sin vt.$$

*) $t_n \in \mathrm{BV}$ means $\sum |t_n - t_{n-1}| < \infty$.

2. Recently PREM CHANDRA [1] has proved the following theorem.

Theorem A. *Let for* $0<\alpha<1$, *the strictly increasing sequences* $\{\lambda_n\}$ *and* $\{g(n)\}$, *of nonnegative terms, tending to infinity with n, satisfy the following conditions:*

$$\log(\Pi/t) = O\{g(k/t)\}; \quad as \quad t \to 0, \tag{2.1}$$

$$\{\lambda_n/(n+1)\}\uparrow \quad with \quad n \geqq n_0, \tag{2.2}$$

$$n^{1-\alpha}\Delta\lambda_n = O\{\lambda_{n+1}\}, \quad as \quad n \to \infty, \tag{2.3}$$

$$\begin{cases} \text{(i)} \quad \{x/g(x)\}\uparrow \quad with \quad x, \\ \text{(ii)} \quad x\dfrac{d}{dx}\left(\dfrac{1}{g(k/x)}\right)\uparrow \quad with \quad x, \\ \text{(iii)} \quad \dfrac{d}{dx}\left(\dfrac{1}{g(k/x)}\right)\downarrow \quad with \quad x, \end{cases} \tag{2.4}$$

$$\begin{cases} \text{(i)} \quad \left[\dfrac{d}{dt}\left(\dfrac{1}{g(k/t)}\right)\right]_{t=1/n} = O\{n/g(n)\}, \\ \text{(ii)} \quad \sum\limits_{n=1}^{\infty}(ng(n))^{-1} < \infty. \end{cases} \tag{2.5}$$

Then, if $\Phi(t)\in\mathrm{BV}(0,\pi)$ *and* $\Lambda(t)g(k/t)\in\mathrm{BV}(0,\pi)$, *the series*

$$\sum_{n=1}^{\infty} A_n(x) \in |R, \lambda_n, 1|,$$

where k is a suitable positive constant such that $g(k/t)>0$ *for* $t>0$.

3. The object of this note is to generalize the above theorem. In what follows we prove the following:

Theorem. *Let, for* $0<\alpha<1$, *the strictly increasing sequences* $\{\lambda_n\}$ *and* $\{g(n)\}$, *of nonnegative terms, tending to infinity with n, satisfy the following conditions:*

$$\log(\pi/t) = O\{g(k/t)\}, \quad as \quad t \to 0, \tag{3.1}$$

$$\{\lambda_n/n^{\delta}\}\uparrow, \quad n \geqq n_0, \quad 0<\alpha<\delta<1, \tag{3.2}$$

$$n^{1-\alpha}\Delta\lambda_n = O\{\lambda_{n+1}\}, \quad as \quad n \to \infty, \tag{3.3}$$

$$\begin{cases} \text{(i)} \quad \{x^{\beta}/g(x)\}\uparrow \quad with \quad x, \beta \geqq 1, \\ \text{(ii)} \quad \dfrac{d}{dx}\left(\dfrac{1}{g(k/x)}\right)\downarrow \quad with \quad x, \end{cases} \tag{3.4}$$

$$\begin{cases} \text{(i)} \quad \left[\dfrac{d}{dt}\dfrac{1}{g(k/t)}\right]_{t=1/n} = O(n/g(n)), \\ \text{(ii)} \quad \sum\limits_{n=1}^{\infty}\dfrac{n^{\beta-2}}{g(n)} < \infty. \end{cases} \tag{3.5}$$

Then, if $\Phi(t)\in \mathrm{BV}\,(0,\pi)$ and $\Lambda(t)g(k/t)\in \mathrm{BV}\,(0,\pi)$, the series

$$\sum_{n=1}^{\infty} A_n(x)\in |R, \lambda_n, 1|,$$

where k is a suitable positive constant such that $g(k/t)>0$ for $t>0$.

Remarks:

I. Condition (2.2)$\Rightarrow$(3.2) but the converse is not true. Thus (3.2) is a lighter condition.

II. For $\beta=1$ we get a result which is a generalization of Theorem A.

III. Condition (2.4) (ii) of PREM CHANDRA is redundant. For he employs this condition in the proof of

$$\int_0^t \sin(n+1)u\frac{d}{du}\left(\frac{1}{g(k/u)}\right)du = O\left\{\frac{1}{g(n+1)}\right\} \tag{3.6}$$

which can be proved in the following manner without the use of the above condition.

PROOF of (3.6): Case I: $(n+1)^{-1}\leqq t$.

$$\int_0^t \sin(n+1)u\frac{d}{du}\left(\frac{1}{g(k/u)}\right)du = \left(\int_0^{(n+1)^{-1}} + \int_{(n+1)^{-1}}^{t}\right)\sin(n+1)u\frac{d}{du}\frac{1}{g(k/u)}\,du =$$
$$= I_1+I_2,\ \text{say}.$$

Since $|\sin(n+1)u|\leqq (n+1)u$,

$$I_1\leqq (n+1)\int_0^{(n+1)^{-1}} u\frac{d}{du}\left(\frac{1}{g(k/u)}\right)du\leqq$$

$$\leqq \int_0^{(n+1)^{-1}}\frac{d}{du}\left(\frac{1}{g(k/u)}\right)du = \left[\frac{1}{g(k/u)}\right]_0^{(n+1)^{-1}} = \frac{1}{g\big(k(n+1)\big)} = O\left(\frac{1}{g(n+1)}\right).$$

And, by virtue of second mean value theorem and the conditions (3.4) (ii), (3.5) (i), we have

$$I_2 = \left\{\frac{d}{du}\left(\frac{1}{g(k/u)}\right)\right\}_{u=(n+1)^{-1}}\int_{(n+1)^{-1}}^{\xi}\sin(n+1)u\,du = O\left\{\frac{1}{g(n+1)}\right\},$$
$$(n+1)^{-1}<\xi<t.$$

Case II: $(n+1)^{-1}>t$. In this case we write,

$$\int_0^t \sin(n+1)u\frac{d}{du}\left(\frac{1}{g(k/u)}\right)du = \left(\int_0^{(n+1)^{-1}} - \int_t^{(n+1)^{-1}}\right)\sin(n+1)u\left(\frac{d}{du}\frac{1}{g(k/u)}\right)du =$$
$$= I_1 - I_2',\ \text{say}.$$

Proceeding as in the case of I_1 it can be easily shown that

$$I_2' = O\left(\frac{1}{g(n+1)}\right).$$

This completes the proof of (3.6).

4. For the proof of our theorem we require the following lemmas:

Lemma 1. *If* $\{\lambda_n/n^\delta\}\uparrow$, $n \geqq n_0$, $0<\delta<1$, *then*

$$K(n, t) = O\left\{\frac{\lambda_n t^{-\delta}}{n^\delta}\right\},$$

uniformly in $0<t\leqq\pi$.

PROOF. By virtue of hypothesis and Abel's Lemma

$$K(n, t) \leqq \frac{\lambda_n}{n^\delta} \operatorname{Max} \left| \sum_{v \geqq n_0} \frac{\sin vt}{(v+1)^{1-\delta}} \right| + O(1).$$

But

$$\sum_{v=n_0}^{m} \frac{\sin vt}{(v+1)^{1-\delta}} = \sum_{v=n_0}^{[1/t]} \frac{\sin vt}{(v+1)^{1-\delta}} + \sum_{[1/t]+1}^{m} \frac{\sin vt}{(v+1)^{1-\delta}} = L_1 + L_2.$$

$$L_1 = \sum_{v=n_0}^{[1/t]} \frac{(v+1)^\delta \sin vt}{(v+1)} \leqq \left(1+\left[\frac{1}{t}\right]\right)^\delta \operatorname{Max} \left| \sum_{v>n_0} \frac{\sin vt}{(v+1)} \right| = O(t^{-\delta}),$$

uniformly in $0 < t \leqq \pi$.

$$L_2 = \sum_{v=[1/t]+1}^{m} \frac{\sin vt}{(v+1)^{1-\delta}} \leqq t^{1-\delta} \operatorname{Max} \left| \sum_{v \geqq [1/t]} \sin vt \right| = O(t^{-\delta}),$$

uniformly in $0 < t \leqq \pi$. Hence

$$K(n, t) = O\left(\frac{\lambda_n t^{-\delta}}{n^\delta}\right).$$

Lemma 2. *If for* $\beta \geqq 1$, $\{x^\beta/g(x)\}\uparrow$ *with* x *and* $g(x)\uparrow\infty$, $x\to\infty$, *then*

$$\int_0^t \frac{\sin (n+1)u}{ug(k/u)}\, du = O\{n^{\beta-1}/g(n+1)\},$$

uniformly in $0<t\leqq\pi$.

PROOF. Case I: $(n+1)^{-1} \leqq t$

$$\int_0^t \frac{\sin (n+1)u}{ug(k/u)}\, du = \Big(\int_0^{(n+1)^{-1}} + \int_{(n+1)^{-1}}^{t} \Big) \frac{\sin (n+1)u}{ug(k/u)}\, du.$$

$$= M_1 + M_2, \text{ say.}$$

Now as shown by Prem Chandra [1, p. 337]

$$M_1 = O\left\{\frac{1}{g(n+1)}\right\}, \quad n \to \infty.$$

And, by virtue of the first hypothesis

$$M_2 = \int_{(n+1)^{-1}}^{t} \frac{(k/u)^\beta}{g(k/u)} \frac{\sin(n+1)u}{k^\beta u^{1-\beta}}\, du = \frac{(n+1)^\beta}{g\big(k(n+1)\big)} \int_{(n+1)^{-1}}^{\xi} u^{\beta-1} \sin(n+1)u\, du =$$

$$\left(\frac{1}{n+1} < \xi < t\right) \qquad = \frac{(n+1)^\beta}{g\big(k(n+1)\big)} \xi^{\beta-1} \int_{\eta}^{\xi} \sin(n+1)u\, du, =$$

$$\big((n+1)^{-1} < \eta < \xi\big) \qquad = O\{n^{\beta-1}/g(n+1)\},$$

uniformly in $0 < t \leqq \pi$.

Case II: $(n+1)^{-1} > t$.

$$\int_0^t \frac{\sin(n+1)u}{ug(k/u)}\, du = \Big(\int_0^{(n+1)^{-1}} - \int_t^{(n+1)^{-1}}\Big) \frac{\sin(n+1)u}{ug(k/u)}\, du = M_1 - M_2'.$$

Now

$$M_2' = \int_t^{(n+1)^{-1}} \frac{\sin(n+1)u}{ug(k/u)}\, du \leqq (n+1) \int_t^{(n+1)^{-1}} \frac{du}{g(k/u)} = O\left(\frac{1}{g(n+1)}\right),$$

uniformly in $0 < t \leqq \pi$.

This completes the proof of Lemma 2.

Proof of Theorem 5. Following Prem Chandra [1] we have

$$A_n(x) = \frac{2}{\pi} \int_0^\pi \Lambda(t) g(k/t) \frac{t}{g(k/t)} \frac{d}{dt}\left(\frac{\sin nt}{nt}\right) dt$$

and the series $\sum_{n=1}^{\infty} A_n(x) \in |R, \lambda_n, 1|$, if

$$\sum = \sum_{n=0}^{\infty} \left| \frac{\Delta\lambda_n}{\lambda_n \lambda_{n+1}} \sum_{v=0}^{n} \frac{\lambda_v}{(v+1)} \int_0^t \frac{u}{g(k/u)} \frac{d}{du}\left(\frac{\sin(v+1)u}{u}\right) du \right| = O(1),$$

uniformly in $0 < t \leqq \pi$.

Now, integrating by parts we have

$$\int_0^t \frac{u}{g(k/u)} \frac{d}{du}\left(\frac{\sin(v+1)u}{u}\right) du = \frac{\sin(v+1)t}{g(k/t)} - \int_0^t \frac{\sin(v+1)u}{ug(k/u)}\, du -$$

$$- \int_0^t \sin(v+1)u \frac{d}{du}\left(\frac{1}{g(k/u)}\right) du.$$

Therefore,

$$\sum \leqq \frac{1}{g(k/t)} \sum_{n=0}^{\infty} \left| \frac{\Delta\lambda_n}{\lambda_n \lambda_{n+1}} K(n,t) \right| + \sum_{n=0}^{\infty} \left| \frac{\Delta\lambda_n}{\lambda_n\lambda_{n+1}} \sum_{v=0}^{n} \frac{\lambda_v}{v+1} \int_0^t \frac{\sin(v+1)u}{ug(k/u)}\, du \right| +$$

$$+ \sum_{n=0}^{\infty} \left| \frac{\Delta\lambda_n}{\lambda_n\lambda_{n+1}} \sum_{v=0}^{n} \frac{\lambda_v}{(v+1)} \int_0^t \sin(v+1)u \frac{d}{du}\left(\frac{1}{g(k/u)}\right) du = \sum_1 + \sum_2 + \sum_3, \text{ say.}$$

Now, we write, for $T = \left[t^{-\frac{\delta}{\delta-\alpha}}\right]$

$$\sum_1 = \sum_{n=0}^{T-1} + \sum_{n=T}^{\infty} = \sum_{1,1} + \sum_{1,2}, \text{ say.}$$

As shown by Prem Chandra [1, p. 339], we have

$$\sum_{1,1} = O(1),$$

uniformly in $0 < t \leqq \pi$.

Now, using Lemma 1, we have

$$\sum_{1,2} = \frac{1}{g(k/t)} \sum_{n \neq T}^{\infty} \left| \frac{\Delta\lambda_n}{\lambda_n\lambda_{n+1}} K(n,t) \right| = O\left\{ \frac{1}{g(k/t)} \sum_{n=T}^{\infty} \frac{|\Delta\lambda_n|}{\lambda_n\lambda_{n+1}} \cdot \frac{\lambda_n t^{-\delta}}{n^\delta} \right\} =$$

$$= O\left\{ \frac{1}{g(k/t)} \sum_{n=T}^{\infty} \frac{\lambda_{n+1}}{n^{1-\alpha}} \cdot \frac{t^{-\delta}}{\lambda_{n+1} n^\delta} \right\} = O\left\{ \frac{t^{-\delta}}{g(k/t)} \sum_{n=T}^{\infty} \frac{1}{n^{1+\delta-\alpha}} \right\} = O(1),$$

uniformly in $0 < t \leqq \pi$. And, by virtue of Lemma 2 and (3.5) (ii)

$$\sum_2 = O\left\{ \sum_{n=0}^{\infty} \frac{|\Delta\lambda_n|}{\lambda_n\lambda_{n+1}} \sum_{v=0}^{n} \lambda_v \cdot \frac{(v+1)^{\beta-2}}{g(v+1)} \right\} =$$

$$= O\left\{ \sum_{v=0}^{\infty} \frac{\lambda_v (v+1)^{\beta-2}}{g(v+1)} \sum_{n=v}^{\infty} \left(\frac{1}{\lambda_n} - \frac{1}{\lambda_{n+1}} \right) \right\} = O\left\{ \sum_{v=0}^{\infty} \frac{(v+1)^{\beta-2}}{g(v+1)} \right\} = O(1),$$

uniformly in $0 < t \leqq \pi$. Also, by using (3.6), we have $\sum_3 = O(1)$, uniformly in $0 < t \leqq \pi$.

This completes the proof of the theorem.

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References

[1] P. Chandra, Absolute summability by Riesz means, *Pacific J. Math.* **34** (1970), 335—341.

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