A number-theoretic identity

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1. An arithmetic function $f(n, r)$ is said to be an 'Even function of n (mod r)' ([1]) if $f(n, r)=f((n, r), r)$ for all values of n and $r\geqq 1$. ECKFORD COHEN [1] has shown that $f(n, r)$ is even (mod r) if and only if it possesses a Fourier Expansion of the form

$$f(n, r) = \sum_{d|r} \alpha(d, r) C(n, d) \tag{1.1}$$

where $C(n, r)$ is *Ramanujan's Trigonometric Sum* defined by

$$C(n, r) = \sum_{\substack{h (\mathrm{mod}\, r) \\ (h, r)=1}} \exp(2\pi i h n/r)$$

and $\alpha(d, r)$ is determined by the formula

$$\alpha(d, r) = \frac{1}{r} \sum_{d|r} f(r/\delta, r) C(r/d, \delta) \tag{1.2}$$

The *Cauchy-Composition* (mod r) [2] of two even functions (mod r) namely $f(n, r)$ and $g(n, r)$ is defined by

$$h(n, r) = \sum_{n\equiv a+b\, (\mathrm{mod}\, r)} f(a, r) g(b, r) \tag{1.3}$$

the summation extending over a, b (mod r) such that $n \equiv a+b$ (mod r).

If $f(n, r)$ has the representation (1.1) and

$$g(n, r) = \sum_{d|r} \beta(d, r) C(n, d) \tag{1.4}$$

then it is known ([2]) that the Cauchy product of f and g has the representation

$$h(n, r) = r \sum_{d|r} \alpha(d, r) \beta(d, r) C(n, d). \tag{1.5}$$

Next, let k be a positive integer. If $\varphi_k(r)$ denotes the number of elements of a k-reduced residue system (mod r), then $\varphi_k(r)$ is given [3] by

$$\varphi_k(r) = \sum_{d|r} \mu(r/d) d^k \tag{1.6}$$

where $\mu(r)$ is the Möbius Function.

For $k=1$ $\varphi_k(r)$ reduces to Euler's φ-function. It can be easily verified that $\varphi(r)$ divides $\varphi_k(r)$.

We introduce the function $S_k(r)$ defined by

$$S_k(r) = \sum_{d|r} \frac{\varphi_k(d)}{\varphi(d)} \tag{1.8}$$

Clearly, for $k=1$, $S_k(r)=d(r)$, the number of divisors of r.

The aim of this note is to derive an identity (see Theorem in 3.) which generalizes the following interesting relation due to KESAVA MENON [4]:

$$\sum_{(a,r)=1} (a-1, r) = \varphi(r)d(r) \tag{1.9}$$

where the summation on the left is over a reduced residue system (mod r).

2. We need the Fourier representations of two even functions $f(n, r)$ and $\varrho(n, r)$ defined below:

Let

$$f(n, r) = (n, r)^k. \tag{2.1}$$

Then,

$$f(n, r) = \sum_{d|r} \alpha(d, r) C(n, d)$$

where

$$\alpha(d, r) = \frac{1}{r} \sum_{\delta|r} \delta^k C(r/d, r/\delta).$$

Since

$$C(n, r) = \sum_{d|(n,r)} \mu(r/d)d$$

$$\alpha(d, r) = \frac{1}{r} \sum_{\delta|r} \delta^k \sum_{D|(r/d,\, r/\delta)} \mu(r/D\delta)D,$$

interchanging the order of summation in the right hand side, we have

$$\alpha(d, r) = \frac{1}{r} \sum_{D|r/d} D \sum_{\delta|r/D} \mu(r/D\delta)\delta^k = \frac{1}{r} \sum_{D|r/d} D\varphi_k(r/D),$$

by (1.6). Or

$$\alpha(d, r) = \frac{1}{r} \sum_{\delta|r/d} \delta\varphi_k(r/\delta) \tag{2.2}$$

Now, let

$$\varrho(n, r) = \begin{cases} 1, & \text{if} \quad (n, r) = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

Then,

$$\varrho(n, r) = \sum_{d|r} \beta(d, r) C(n, d)$$

where

$$\beta(d, r) = \frac{1}{r} \frac{\mu(d)\varphi(r)}{\varphi(d)}$$

3. Theorem. *If* $(n, r) = 1$,

$$\sum_{(a,r)=1} (|n-a|, r)^k = \varphi(r) s_k(r).$$

Proof. We note that $\sum_{(a,r)=1} (|n-a|, r)^k$ is the Cauchy product (mod r) of $f(n, r)$ (2.1) and $\varrho(n, r)$ (2.3). Therefore, by (1.5)

$$\sum_{(a,r)=1} (|n-a|, r)^k = \frac{\varphi(r)}{r} \sum_{d|r} \left\{ \sum_{\delta|r/d} \delta \varphi_k\left(\frac{r}{\delta}\right) \right\} \frac{\mu(d)}{\varphi(d)} C(n, d).$$

As $(n, r)=1$, $(n, d)=1$ for every divisor d of r. So, $C(n, d)=\mu(d)$. The sum on the right, then, reduces to

$$\frac{\varphi(r)}{r} \sum_{d|r} \left\{ \sum_{\delta|r/d} \delta \varphi_k\left(\frac{r}{\delta}\right) \right\} \frac{\mu^2(d)}{\varphi(d)} = \frac{\varphi(r)}{r} \sum_{\delta|r} \delta \varphi_k\left(\frac{r}{\delta}\right) \sum_{d|r/\delta} \frac{\mu^2(d)}{\varphi(d)}.$$

As

$$\sum_{d|r} \frac{\mu^2(d)}{\varphi(d)} = \frac{r}{\varphi(r)},$$

further simplification leads to

$$\sum_{(a,r)=1} (|n-a|, r)^k = \varphi(r) \sum_{\delta|r} \frac{\varphi_k\left(\frac{r}{\delta}\right)}{\left(\frac{r}{\delta}\right)} \frac{\left(\frac{r}{\delta}\right)}{\varphi\left(\frac{r}{\delta}\right)} = \varphi(r) \sum_{d|r} \frac{\varphi_k(d)}{\varphi(d)} = \varphi(r) s_k(r), \text{ by (18.)}$$

This completes the proof of the Theorem.

Corollary: (1.9) is the special case of the above theorem for $k=1$, $n=1$.

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References

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