# On some pairwise normality conditions in bitopological spaces

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J. C. KELLY [3] initiated a study of bitopological spaces. A set $X$ equipped with two topologies is called a bitopological space. Separation axioms in bitopological spaces have been studied by various authors, e. g., J. C. KELLY [3], E. P. LANE [4], C. W. PATTY [6], M. G. MURDESHWAR and S. A. NAIMPALLY [5], M. K. SINGAL and ASHA RANI [9], ASHA RANI and SHASHI PRABHA ARYA [7]. The concepts of normality and complete regularity in bitopological spaces were first introduced by J. C. KELLY [3]. The purpose of the present paper is to introduce the concepts of almost normal, mildly normal and almost completely regular spaces in bitopological spaces and to study some of their consequences. These notions for topological spaces have been introduced and studied in [8, 10].

Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be a bitopological space. A subset $A$ of $X$ is said to be $(i, j)$-regularly open if $A=\mathscr{T}_i\text{-int}\ \mathscr{T}_j\text{-cl}\ A$, $i, j=1, 2$, $i\neq j$. It is easy to see that a set is $(i, j)$-regularly open if and only if it is the $\mathscr{T}_i$-interior of a $\mathscr{T}_j$-closed set, $i, j=1, 2$, $i\neq j$. $A$ is said to be $(i, j)$-regularly closed if $A=\mathscr{T}_i\text{-cl}\ \mathscr{T}_j\text{-int}\ A$, $i, j=1, 2$, $i\neq j$. A set is $(i, j)$-regularly closed if and only if it is the $\mathscr{T}_i$-closure of a $\mathscr{T}_j$-open set, $i, j=1, 2$, $i\neq j$. Also, a set is $(i, j)$-regularly closed if and only if its complement is $(i, j)$-regularly open $i, j=1, 2$, $i\neq j$.

## 1. Pairwise almost normal spaces

*Definition 1.1.* A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be *pairwise almost normal* if for every pair of disjoint sets $A$ and $B$ such that $A$ is $\mathscr{T}_i$-closed, $B$ is $(j, i)$-regularly closed, there exists a $\mathscr{T}_j$-open set $U$ and a disjoint $\mathscr{T}_i$-open set $V$ such that $A\subseteq U$, $B\subseteq V$, $i, j=1, 2$, $i\neq j$.

Obviously, every pairwise normal space is pairwise almost normal and every pairwise almost normal, bi-$T_1$ space is pairwise almost regular [7].

**Theorem 1.1.** *For a space $(X, \mathscr{T}_1, \mathscr{T}_2)$ the following are equivalent,*

(a) *$(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise almost normal.*

(b) *For every $\mathscr{T}_i$-closed set $A$ and every $(j, i)$-regularly open set $B$ such that $A\subseteq B$ there exists a $\mathscr{T}_j$-open set $V$ such that $A\subseteq V\subseteq \mathscr{T}_i\text{-cl}\ V\subseteq B$.*

(c) *For every $(i, j)$-regularly closed set $A$ and every $\mathscr{T}_j$-open set $B$ such that $A\subseteq B$, there exists a $\mathscr{T}_j$-open set $V$ such that $A\subseteq V\subseteq \mathscr{T}_i\text{-cl}\ V\subseteq B$.*

(d) *For every pair of disjoint sets $A$ and $B$ such that $A$ is $\mathscr{T}_i$-closed, $B$ is $(j, i)$-regularly closed, there exist disjoint sets $U$ and $V$ such that $A\subseteq U$, $B\subseteq V$, $U$ is $\mathscr{T}_j$-open, $V$ is $\mathscr{T}_i$-open and $\mathscr{T}_i\text{-cl}\ U\cap \mathscr{T}_j\text{-cl}\ V = \emptyset$.*

PROOF. $(a) \Rightarrow (b)$. If $A$ be a $\mathcal{T}_i$-closed set and $B$ be a $(j, i)$-regularly open set such that $A \subseteq B$, then $A$ and $X \sim B$ are disjoint sets such that $A$ is $\mathcal{T}_i$-closed, $X \sim B$ is $(j, i)$-regularly closed. Therefore there exists a $\mathcal{T}_j$-open set $V$ and a disjoint $\mathcal{T}_i$-open set $U$ such that $A \subseteq V$, $X \sim B \subseteq U$. Now, $U \cap V = \emptyset \Rightarrow U \cap \mathcal{T}_i\text{-cl } V = \emptyset$. Thus, $A \subseteq V \subseteq \mathcal{T}_i\text{-cl } V \subseteq X \sim U \subseteq B$.

$(b) \Rightarrow (c)$. If $A$, $B$ be as given in (c), then $X \sim A$ is an $(i, j)$-regularly open set containing the $\mathcal{T}_j$-closed set $X \sim B$. Therefore there exists a $\mathcal{T}_i$-open set $U$ such that $X \sim B \subseteq U \subseteq \mathcal{T}_j\text{-cl } U \subseteq X \sim A$. Now let $V = X \sim \mathcal{T}_j\text{-cl } U$. Then $V$ is a $\mathcal{T}_j$-open set such that $A \subseteq V \subseteq \mathcal{T}_i\text{-cl } V \subseteq \mathcal{T}_i\text{-cl } (X \sim U) = X \sim U \subseteq B$.

$(c) \Rightarrow (d)$. With $A$ and $B$ as given in (d), $X \sim A$ is a $\mathcal{T}_i$-open set containing the $(j, i)$-regularly closed set $B$. Therefore, there exists a $\mathcal{T}_i$-open set $W$ such that $B \subseteq W \subseteq \mathcal{T}_j\text{-cl } W \subseteq X \sim A$. Again there exists a $\mathcal{T}_i$-open set $W^*$ such that $B \subseteq W^* \subseteq \mathcal{T}_2\text{-cl } W^* \subseteq W \subseteq \mathcal{T}_2\text{-cl } W \subseteq X \sim A$. Let $V = W^*$, $U = X \sim \mathcal{T}_2\text{-cl } W$. Then $A \subseteq U$, $B \subseteq V$, $U$ is $\mathcal{T}_j$-open, $V$ is $\mathcal{T}_i$-open. Now, $U \subseteq X \sim W$. Therefore, $\mathcal{T}_i\text{-cl } U \subseteq \mathcal{T}_i\text{-cl } (X \sim W) = X \sim W \subseteq X \sim \mathcal{T}_j\text{-cl } W^*$. Hence, we have, $\mathcal{T}_i\text{-cl } U \cap \mathcal{T}_j\text{-cl } W^* = \emptyset$.

$(d) \Rightarrow (a)$. Obvious.

*Remark 1.1.* The set $V$ in characterizations (b) and (c) above may be taken to be $(j, i)$-regularly open.

*Definition 1.2.* $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise semi-normal* if for every $(i, j)$-regularly open set $G$ containing a $\mathcal{T}_j$-closed set $F$, there exists a $\mathcal{T}_i$-open set $V$ such that $F \subseteq V \subseteq \mathcal{T}_i\text{-int } \mathcal{T}_j\text{-cl } V \subseteq G$.

Obviously, every pairwise normal space is pairwise semi-normal and every pairwise-normal bi-$T_1$ space is pairwise semi-regular [7].

**Theorem 1.2.** *$(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise normal if and only if $X$ is pairwise almost normal and pairwise semi-normal.*

PROOF. Only the 'if' part need be proved. Let $A$ be $\mathcal{T}_i$-closed and let $B$ be a $\mathcal{T}_j$-open set containing $A$. Since $X$ is pairwise semi-normal, therefore there exists a $\mathcal{T}_j$-open set $V$ such that $A \subseteq V \subseteq \mathcal{T}_j\text{-int } \mathcal{T}_i\text{-cl } V \subseteq B$. Since $X$ is pairwise almost normal, therefore there exists a $\mathcal{T}_j$-open set $W$ such that $A \subseteq W \subseteq \mathcal{T}_i\text{-cl } W \subseteq \mathcal{T}_j\text{-int } \mathcal{T}_i\text{-cl } V \subseteq B$. Hence $X$ is pairwise normal.

**Theorem 1.3.** *$(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise almost normal if and only if for every pair of disjoint sets $A$ and $B$ such that $A$ is $\mathcal{T}_i$-closed, $B$ is $(j, i)$-regularly closed, there exists a real-valued function $g: X \to [0, 1]$ such that $g(A) = \{0\}$, $g(B) = \{1\}$ and $g$ is $\mathcal{T}_j$-upper semicontinuous and $\mathcal{T}_i$-lower semicontinuous, $i, j = 1, 2$, $i \neq j$.*

PROOF. To prove the 'if' part, let $A$ and $B$ be any two disjoint sets such that $A$ is $\mathcal{T}_i$-closed and $B$ is $(j, i)$-regularly closed. Then there exists a $\mathcal{T}_j$-upper semicontinuous and $\mathcal{T}_i$-lower semicontinuous function $g: X \to [0, 1]$ such that $g(A) = \{0\}$, $g(B) = \{1\}$. If $U = g^{-1}([0, 1/2[)$ and $V = g^{-1}(]1/2, 1])$, then $U \cap V = \emptyset$, and $U$ is a $\mathcal{T}_j$-open set containing $A$ and $V$ is a $\mathcal{T}_i$-open set containing $B$. Hence $X$ is pairwise almost normal. Conversely, let $A$ and $B$ be sets as given. Let $G_0 = A$, $K_1 = X \sim B$. Then $G_0$ is $\mathcal{T}_i$-closed, $K_1$ is $(j, i)$-regularly open such that $G_0 \subseteq K_1$. Since $X$ is pairwise almost normal, therefore there exists a $(j, i)$-regularly open set $K_{1/2}$ and an $(i, j)$-regularly closed set $G_{1/2}$ such that $G_0 \subseteq K_{1/2} \subseteq G_{1/2} \subseteq K_1$. Again, by pairwise almost normality, there exist $(j, i)$-regularly open sets $K_{1/4}$, $K_{3/4}$ and $(i, j)$-regularly closed sets $G_{1/4}$, $G_{3/4}$ such that $G_0 \subseteq K_{1/4} \subseteq G_{1/4} \subseteq K_{1/2} \subseteq G_{1/2} \subseteq K_{3/4} \subseteq G_{3/4} \subseteq K_1$. Continuing this process, we obtain two families $\{G_s\}$ and $\{K_s\}$, where $s=p/2^q$ $(p=1, 2, \ldots, 2^q-1,\ q=1, 2, \ldots)$. If $s$ is any other dyadic rational, let $K_s=\emptyset$ $(s \leqq 0)$, $K_s=X$ $(s>1)$; and $G_s=\emptyset$ $(s<0)$, $G_s=X$ $(s \geqq 1)$. Then, $K_r \subseteq K_s \subseteq G_s \subseteq G_t$ $(r \leqq s \leqq t)$ and $G_s \subseteq K_t$ $(s<t)$. Let $g: X \to [0, 1]$ be a function defined as

$$g(x) = \inf\{t : x \in K_t\} \quad \text{if} \quad x \in K_1$$

and

$$g(x) = 1 \quad \text{if} \quad x \in X \sim K_1.$$

Then, $g(A)=\{0\}$, $g(B)=\{1\}$ and it can be verified as in the proof of Urysohn's Lemma for topological spaces, that $g$ is $\mathcal{T}_j$-upper semicontinuous and $\mathcal{T}_i$-lower semicontinuous.

**Theorem 1.4.** *Every pairwise closed, pairwise continuous and pairwise open image of a pairwise almost normal space is pairwise almost normal.*

PROOF. Let $f$ be a pairwise closed, pairwise continuous and pairwise open mapping of a pairwise almost normal space $(X, \mathcal{T}_1, \mathcal{T}_2)$ onto a space $(Y, \mathcal{T}_1^*, \mathcal{T}_2^*)$. Let $A$ be a $\mathcal{T}_i^*$-closed subset of $Y$ and let $B$ be a $(j, i)$-regularly open subset of $Y$ containing $A$. Then $f^{-1}(A)$ is a $\mathcal{T}_i$-closed subset of $X$ contained in the $\mathcal{T}_j$-open subset $f^{-1}(B)$ of $X$. Since $f^{-1}(B)$ is $\mathcal{T}_j$-open, therefore $f^{-1}(B) \subseteq \mathcal{T}_j\text{-int}\, \mathcal{T}_i\text{-cl}\,(f^{-1}(B))$. Thus, $\mathcal{T}_j\text{-int}\, \mathcal{T}_i\text{-cl}\,(f^{-1}(B))$ is a $(j, i)$-regularly open subset of $X$ containing the $\mathcal{T}_i$-closed set $f^{-1}(A)$. Since $X$ is pairwise normal, therefore, there exists a $(j, i)$-regularly open subset $U$ of $X$ such that $f^{-1}(A) \subseteq U \subseteq \mathcal{T}_i\text{-cl}\, U \subseteq \mathcal{T}_j\text{-int}\, \mathcal{T}_i\text{-cl}\, f^{-1}(B)$. Since $f$ is pairwise closed and pairwise continuous, therefore $\mathcal{T}_i\text{-cl}\, f^{-1}(B) = f^{-1}(\mathcal{T}_i^*\text{-cl}\, B)$. Also, since $f$ is pairwise open and pairwise continuous, therefore $\mathcal{T}_j\text{-int}\, f^{-1}(\mathcal{T}_i^*\text{-cl}\, B) = f^{-1}(\mathcal{T}_j^*\text{-int}\, \mathcal{T}_i^*\text{-cl}\, B)$. Thus $f^{-1}(A) \subseteq U \subseteq \mathcal{T}_i\text{-cl}\, U \subseteq f^{-1}(\mathcal{T}_j^*\text{-int}\, \mathcal{T}_i^*\text{-cl}\, B)$. Hence $A \subseteq f(U) \subseteq f(\mathcal{T}_i\text{-cl}\, U) \subseteq \mathcal{T}_j^*\text{-int}\, \mathcal{T}_i^*\text{-cl}\, B = B$. Since $f$ is pairwise open, therefore $f(U)$ is $\mathcal{T}_j^*$-open and since $f$ is pairwise closed and pairwise continuous, therefore $f(\mathcal{T}_i\text{-cl}\, U) = \mathcal{T}_i^*\text{-cl}\,(f(U))$. Thus $A \subseteq f(U) \subseteq \mathcal{T}_i^*\text{-cl}\, f(U) \subseteq B$. Hence $Y$ is pairwise almost normal.

**Theorem 1.5.** *Every bi-clo-open subspace of a pairwise almost normal space is pairwise almost normal.*

PROOF. Let $(Y, \mathcal{T}_{1y}, \mathcal{T}_{2y})$ be a subspace of a pairwise almost normal space $(X, \mathcal{T}_1, \mathcal{T}_2)$ which is both $\mathcal{T}_1$-clo-open and $\mathcal{T}_2$-clo-open. Let $A$ be a $\mathcal{T}_{iy}$-closed subset of $Y$ and let $B$ be a relatively $(j, i)$-regularly open subset of $Y$ containing $A$. Since $Y$ is $\mathcal{T}_i$-closed, therefore $A$ is $\mathcal{T}_i$-closed. Also, it may be easily verified that $B = \mathcal{T}_j\text{-int}\, \mathcal{T}_i\text{-cl}\, B$. Thus $B$ is a $(j, i)$-regularly open subset of $X$ containing the $\mathcal{T}_i$-closed set $A$. Since $X$ is pairwise almost normal, therefore there exists a $\mathcal{T}_j$-open set $U$ such that $A \subseteq U \subseteq \mathcal{T}_i\text{-cl}\, U \subseteq B$. Then $U$ is a $\mathcal{T}_{jy}$-open subset of $Y$ such that $A \subseteq U \subseteq \mathcal{T}_{iy}\text{-cl}\, U \subseteq B$. Hence $Y$ is pairwise almost normal.

## 2. Pairwise mildly normal spaces

*Definition 2.1.* A space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be *pairwise mildly normal* if for every pair of disjoint sets $A$ and $B$ such that $A$ is $(i, j)$-regularly closed, $B$ is $(j, i)$-regularly closed, there exist disjoint sets $U$ and $V$ such that $A \subseteqq U$, $B \subseteqq V$, $U$ is $\mathscr{T}_j$-open and $V$ is $\mathscr{T}_i$-open, $i, j = 1, 2$, $i \neq j$.

Obviously, every pairwise almost normal space is pairwise mildly normal.

**Theorem 2.1.** *For a space $(X, \mathscr{T}_1, \mathscr{T}_2)$, the following are equivalent:*

(a) *$X$ is pairwise mildly normal.*

(b) *For every $(i, j)$-regularly closed set $A$ and every $(j, i)$-regularly open set $B$ such that $A \subseteqq B$, there exists a $\mathscr{T}_j$-open set $V$ such that $A \subseteqq V \subseteqq \mathscr{T}_i$-cl $V \subseteqq B$.*

(c) *For every $(i, j)$-regularly closed set $A$ and every $(j, i)$-regularly closed set $B$ such that $A \cap B = \emptyset$, there exist disjoint sets $U$ and $V$ such that $A \subseteqq U$, $B \subseteqq V$, $U$ is $\mathscr{T}_j$-open, $V$ is $\mathscr{T}_i$-open and $\mathscr{T}_i$-cl $U \cap \mathscr{T}_j$-cl $V = \emptyset$.*

PROOF. Similar to the proof of theorem 1.1.

*Remark 2.1.* The set $V$ in characterisation (b) above may be taken to be $(j, i)$-regularly open.

**Theorem 2.2.** *$(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise mildly normal if and only if for every pair of disjoint sets $A$ and $B$ such that $A$ is $(i, j)$-regularly closed, $B$ is $(j, i)$-regularly closed, there exists a real-valued function $g: X \to [0, 1]$ such that $g(A) = \{0\}$, $g(B) = \{1\}$ and $g$ is $\mathscr{T}_j$-upper semicontinuous and $\mathscr{T}_i$-lower semicontinuous.*

PROOF. Similar to the proof of theorem 1.3.

*Definition 2.2.* Let $f$ be a real-valued function on a space $(X, \mathscr{T}_1, \mathscr{T}_2)$. Let $f^{*i}$ and $f_{*i}$, $i = 1, 2$ be functions defined as $f^{*i}(x) = \inf \{\sup_{y \in N} f(y) : N$ is a $\mathscr{T}_i$-neighbourhood of $x\}$ and $f_{*i}(x) = \sup \{\inf_{y \in N} (f(y) : N$ is a $\mathscr{T}_i$-neighbourhood of $x\}$. Then $f$ is $\mathscr{T}_i$-upper semicontinuous if and only if $f^{*i} = f$ and $f$ is $\mathscr{T}_i$-lower semicontinuous if and only if $f_{*i} = f$. Also, $f_{*i}$ is $\mathscr{T}_i$-lower semicontinuous and $f^{*i}$ is $\mathscr{T}_i$-upper semicontinuous. We say that a *$\mathscr{T}_i$-upper semicontinuous function $f$* is $\mathscr{T}_j$-normal if we have $(f_{*j})^{*i} = f$. Also, a *$\mathscr{T}_i$-lower semicontinuous function $f$ is $\mathscr{T}_j$-normal* if $(f^{*j})_{*i} = f$.

**Lemma 2.1.** *If a $\mathscr{T}_i$-upper semicontinuous function $f$ on $(X, \mathscr{T}_1, \mathscr{T}_2)$ is $\mathscr{T}_j$-normal then for each real $\lambda$, $\{x : f(x) > \lambda\}$ is a union of $(i, j)$-regularly closed subsets of $X$.*

PROOF. Let $A = \{x : f(x) > \lambda\}$. Let $x_0 \in A$. Then $f(x_0) \geqq \lambda$. Hence $f(x_0) > \lambda + \delta$ for some $\delta > 0$. Let $B_{x_0} = \{x : f_{*j}(x) > \lambda + \delta\}$. Since $f_{*j}$ is $\mathscr{T}_j$-lower semicontinuous, therefore $B_{x_0}$ is $\mathscr{T}_j$-open. Let $N$ be any arbitrary $\mathscr{T}_i$-open set containing $x_0$. Then $\sup_{y \in N} f_{*j}(y) \geqq (f_{*j})^{*i}(x_0) = f(x_0) > \lambda + \delta$. Hence, $f_{*j}(y) > \lambda + \delta$ for some $y \in N$. Thus, $B_{x_0} \cap N \neq \emptyset$ for all $\mathscr{T}_i$-open sets $N$ containing $x_0$. Therefore $x_0 \in \mathscr{T}_i$-cl $B_{x_0}$. Now, if $y_0 \in \mathscr{T}_i$-cl $B_{x_0}$, then $B_{x_0} \cap N \neq \emptyset$ for every $\mathscr{T}_i$-open set $N$ containing $y_0$. Therefore, $\sup f_{*j}(y) > \lambda + \delta$ for each $\mathscr{T}_i$-open set $N$ containing $y_0$. It follows that $f(y_0) = (f_{*j})^{*i}(y_0) \geqq \lambda + \delta > \lambda$ and therefore $y_0 \in A$. Thus, $\mathscr{T}_i$-cl $B_{x_0} \subseteqq A$. Hence $A \subseteqq \bigcup_{x_0 \in A} \mathscr{T}_i\text{-cl } B_{x_0} \subseteqq A$, that is, $A = \bigcup_{x_0 \in A} \mathscr{T}_i\text{-cl } B_{x_0}$ where each $B_{x_0}$ is $\mathscr{T}_j$-open. Thus $A$ is a union of $(i, j)$-regularly closed subsets of $X$.

*Remark 2.2.* It may be proved similarly that if a $\mathcal{T}_i$-lower semicontinuous function $f$ on $(X, \mathcal{T}_1, \mathcal{T}_2)$ is $\mathcal{T}_j$-normal, then for each real $\lambda$, $\{x: f(x)<\lambda\}$ is a union of $(i, j)$-regularly closed subsets of $X$.

**Lemma 2.2.** *The $\mathcal{T}_i$-closure of every union of $(i, j)$-regularly closed subsets of a space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is $(i, j)$-regularly closed and the $\mathcal{T}_i$-interior of every intersection of $(i, j)$-regularly open sets is $(i, j)$-regularly open.*

PROOF. We shall prove the first assertion and the second will follow by complementation. Let $A = \bigcup_{\alpha\in\Lambda} \mathcal{T}_i\text{-cl}\ \mathcal{T}_j\text{-int}\ A_\alpha$. We want to prove that $\mathcal{T}_i$-cl $A$ is $(i, j)$-regularly closed, that is, $\mathcal{T}_i\text{-cl}\ A = \mathcal{T}_i\text{-cl}\ \mathcal{T}_j\text{-int}\,(\mathcal{T}_i\text{-cl}\ A)$. Now, $\mathcal{T}_j\text{-int}\ \mathcal{T}_i\text{-cl}\ A \subseteq \mathcal{T}_i\text{-cl}\ A$. Therefore $\mathcal{T}_i\text{-cl}\ \mathcal{T}_j\text{-int}\,(\mathcal{T}_i\text{-cl}\ A) \subseteq \mathcal{T}_i\text{-cl}\ A$. Thus, we need only prove that $\mathcal{T}_i\text{-cl}\ A \subseteq \mathcal{T}_i\text{-cl}\ \mathcal{T}_j\text{-int}\,(\mathcal{T}_i\text{-cl}\ A)$. Now, let $x \in \mathcal{T}_i\text{-cl}\ A$ and let $M$ be any $\mathcal{T}_i$-open set containing $x$. Then $M \cap A \neq \emptyset$. This means that there exists an $\alpha \in \Lambda$ such that $M \cap \mathcal{T}_i\text{-cl}\ \mathcal{T}_j\text{-int}\ A_\alpha \neq \emptyset$ But $M$ being $\mathcal{T}_i$-open, this implies that $M \cap \mathcal{T}_j\text{-int}\ A_\alpha \neq \emptyset$, that is,

$$M \cap \Big(\bigcup_{\alpha\in\Lambda} \mathcal{T}_j\text{-int}\ A_\alpha\Big) \neq \emptyset.$$

But $\mathcal{T}_j\text{-int}\ \mathcal{T}_i\text{-cl}\ A \supseteq \bigcup_{\alpha\in\Lambda} \mathcal{T}_j\text{-int}\ A_\alpha$, because,

$$\mathcal{T}_j\text{-int}\ \mathcal{T}_i\text{-cl}\ A = \mathcal{T}_j\text{-int}\ \mathcal{T}_i\text{-cl}\Big(\bigcup_{\alpha\in\Lambda} \mathcal{T}_i\text{-cl}\ \mathcal{T}_j\text{-int}\ A_\alpha\Big) \supseteq$$

$$\supseteq \mathcal{T}_j\text{-int}\Big(\bigcup_{\alpha\in\Lambda} (\mathcal{T}_i\text{-cl}\ \mathcal{T}_j\text{-int}\ A_\alpha)\Big) \supseteq \mathcal{T}_j\text{-int}\Big(\bigcup_{\alpha\in\Lambda} \mathcal{T}_j\text{-int}\ A_\alpha\Big) = \bigcup_{\alpha\in\Lambda} \mathcal{T}_j\text{-int}\ A_\alpha.$$

Thus every $\mathcal{T}_i$-open set $M$ containing $x$ intersects $\mathcal{T}_j\text{-int}\ \mathcal{T}_i\text{-cl}\ A$. Therefore $x \in \mathcal{T}_i\text{-cl}\,(\mathcal{T}_j\text{-int}\ \mathcal{T}_i\text{-cl}\ A)$, that is, $\mathcal{T}_i\text{-cl}\ A \subseteq \mathcal{T}_i\text{-cl}\ \mathcal{T}_j\text{-int}\,(\mathcal{T}_i\ \text{cl}\ A)$. Hence $\mathcal{T}_i\text{-cl}\ A = \mathcal{T}_i\text{-cl}\ \mathcal{T}_j\text{-int}\,(\mathcal{T}_i\text{-cl}\ A)$ which shows that $\mathcal{T}_i$-cl $A$ is an $(i, j)$-regularly closed subset of $X$.

*Definition 2.3.* [KATĚTOV, 1, 2]. Let $R$ and $T$ be sets and let $\varrho$, $\mathcal{T}$ be binary relations defined in $R$, $T$ respectively. Let $\varrho^{\mathcal{T}}$ denote the binary relation in $R^T$ defined by $f_1 \varrho^{\mathcal{T}} f_2$ if and only if $t_1 \mathcal{T} t_2$ implies $f_1(t_1) \varrho f_2(t_2)$, where $f_1, f_2 \in R^T$ and $t_1, t_2 \in T$. The binary relation $\varrho$ in $R$ is said to possess the interpolation property if, given finite subsets $A$ and $B$ of $R$ such that $A \times B \subseteq \varrho$ there exists $c$ in $R$ such that $A \times \{c\} \subseteq \varrho$ and $\{c\} \times B \subseteq \varrho$. If $\varrho$ is a binary relation in $R$, then a binary relation $\bar{\varrho}$ in $R$ is defined as follows: $(x, y) \in \bar{\varrho}$ if and only if for any $u, v \in R$, (i) $(y, v) \in \varrho$ implies $(x, v) \in \varrho$ and (ii) $(u, x) \in \varrho$ implies $(u, y) \in \varrho$.

**Theorem 2.3.** *If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise mildly normal, then for every pair of real-valued functions $f$ and $g$ on $X$ such that $f$ is $\mathcal{T}_1$-lower semicontinuous and $\mathcal{T}_2$-normal and $g$ is $\mathcal{T}_2$-upper semicontinuous and $\mathcal{T}_1$-normal such that $g \leq f$, there exists a $\mathcal{T}_1$-lower semicontinuous and $\mathcal{T}_2$-upper semicontinuous function $h$ on $X$ such that $g \leq h \leq f$.*

PROOF. Define a binary relation $\varrho$ in the power set of $X$ by $A \varrho B$ if $\mathcal{T}_1\text{-cl}\ A \subseteq F \subseteq G \subseteq \mathcal{T}_2\text{-int}\ B$ for some $(1, 2)$-regularly closed set $F$ and some $(2, 1)$-regularly open set $G$. We shall verify that $\varrho$ satisfies conditions 1°, 2° and 3° of Lemma 1 in [9]:

1. Suppose $\{A_i : i=1, \dots, m\} \varrho \{B_j : j=1, \dots, n\}$. For each $i$ and $j$, there exists a (1, 2)-regularly closed set $F_{ij}$ and a (2, 1)-regularly open set $G_{ij}$ such that $\mathcal{T}_1\text{-cl } A_i \subseteq \subseteq F_{ij} \subseteq G_{ij} \subseteq \mathcal{T}_2\text{-int } B_j$. Since $X$ is pairwise mildly normal, therefore there exist (2, 1)-regularly open sets $H_{ij}$ such that $F_{ij} \subseteq H_{ij} \subseteq \mathcal{T}_1\text{-cl } H_{ij} \subseteq \bigcap_j G_{ij}$. For each $i$, $\mathcal{T}_1\text{-cl } A_i \subseteq \bigcap_j F_{ij} \subseteq \bigcap_j H_{ij} \subseteq \bigcap_j \mathcal{T}_1\text{-cl } H_{ij} \subseteq \bigcap_j G_{ij} \subseteq \bigcap_j \mathcal{T}_2\text{-int } B_j$. If $K_i = \mathcal{T}_1\text{-cl} \bigcap_j H_{ij}$ and $M_i = \bigcap_j G_{ij}$, then $K_i$ is (1, 2)-regularly closed and $M_i$ is (2, 1)-regularly open such that $K_i \subseteq M_i$. Again, since $X$ is pairwise mildly normal, there exist (2, 1)-regularly open sets $P_i$ such that $K_i \subseteq P_i \subseteq \mathcal{T}_1\text{-cl } P_i \subseteq M_i$. Thus $\bigcup_i \mathcal{T}_1\text{-cl } A_i \subseteq \bigcup_i K_i \subseteq \subseteq \bigcup_i P_i \subseteq \bigcup_i \mathcal{T}_1\text{-cl } P_i \subseteq \bigcup_i \mathcal{T}_2\text{-int } B_j$. Then $F = \bigcup_i K_i$ is (1, 2)-regularly closed, $G = \mathcal{T}_2\text{-int} \bigcup_i \mathcal{T}_1\text{-cl } P_i$ is (1, 2)-regularly open and for each $i$, $j$, $\mathcal{T}_1\text{-cl } A_i \subseteq F \subseteq G \subseteq \subseteq \mathcal{T}_2\text{-int } B_j$. Again, since $X$ is pairwise mildly normal, there exists a (2, 1)-regularly open set $C$ such that $F \subseteq C \subseteq \mathcal{T}_1\text{-cl } C \subseteq G$. Thus $A_i \varrho C$ and $C \varrho B_j$ for each $i$ and $j$ and therefore $X$ possesses the interpolation property.

2. We shall prove first that $A \subseteq B$ implies $A \varrho B$. If $B \varrho V$, then $\mathcal{T}_1\text{-cl } B \subseteq F \subseteq G \subseteq \subseteq \mathcal{T}_2\text{-int } V$ where $F$ is (1, 2)-regularly closed and $G$ is (2, 1)-regularly open. Since $\mathcal{T}_1\text{-cl } A \subseteq \mathcal{T}_1\text{-cl } B$, therefore $A \varrho V$. Similarly $U \varrho A \Rightarrow U \varrho B$.

3. Obviously, $A \varrho B \Rightarrow A \subseteq B$.

Therefore by Lemma 1 of [4], $\varrho$ possesses the following properties:

$\mathfrak{P}_1$: If $\mathfrak{M}$ and $\mathfrak{N}$ are countable collections of subsets of $X$ and if there exist subsets $A$ and $B$ of $X$ such that $\mathfrak{M} \bar{\varrho} A$, $A \varrho \mathfrak{N}$, $B \bar{\varrho} \mathfrak{N}$, then there exists a subset $C$ of $X$ such that $\mathfrak{M} \varrho C$ and $C \varrho \mathfrak{N}$.

$\mathfrak{P}_2$: For any finite collection $\mathfrak{A}$ of subsets of $X$ there exist subsets $A$ and $B$ of $X$ such that (i) $\mathfrak{A} \bar{\varrho} A$ and $A \varrho U$ whenever $\mathfrak{A} \varrho U$ and (ii) $B \bar{\varrho} \mathfrak{A}$ and $U \varrho B$ whenever $U \varrho \mathfrak{A}$.

Now, let $\mathcal{T}$ be the relation of natural order in the set $Q$ of rational numbers (that is, $t_1 \mathcal{T} t_2$ if $t_1 < t_2$). For $t$ in $Q$, let

$$F(t) = \{x \in X : f(x) < t\},$$

and

$$G(t) = \{x \in X : g(x) \leqq t\}.$$

Since $f$ is $\mathcal{T}_1$-lower semicontinuous and $\mathcal{T}_2$-normal, therefore $F(t)$ is a union of (1, 2)-regularly closed sets in view of Lemma 2.1. Again, since $g$ is $\mathcal{T}_2$-upper semicontinuous and $\mathcal{T}_1$-normal, therefore $X \sim G(t)$ is a union of (2, 1)-regularly closed sets. By Lemma 2.2 then it follows that $\mathcal{T}_1\text{-cl } F(t)$ is (1, 2)-regularly closed and $\mathcal{T}_2\text{-int } G(t)$ is (2, 1)-regularly open. If $t_1, t_2 \in Q$ and $t_1 \mathcal{T} t_2$, then,

$$\mathcal{T}_1\text{-cl } F(t_1) \subseteq \{x \in X : f(x) \leqq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq \mathcal{T}_2\text{-int } G(t_2)$$

and therefore $t_1 \mathcal{T} t_2 \Rightarrow F(t_1) \varrho G(t_2)$. We have thus defined mappings $F$ and $G$ from $Q$ into the power set of $X$ such that $F \varrho^{\mathcal{T}} G$. Because $\varrho$ satisfies properties $\mathfrak{P}_1$ and $\mathfrak{P}_2$ therefore there exists by Lemma 2 of [9] a mapping $R$ from $Q$ into the power set of $X$ such that $F \varrho^{\mathcal{T}} H$, $H \varrho^{\mathcal{T}} H$ and $H \varrho^{\mathcal{T}} G$. For $x \in X$, let $h(x) = \inf \{t \in Q : x \in H(t)\}$. As in the proof of theorem 2.5 in [4] it may be verified that $g \leqq h \leqq f$ and that $h$ is $\mathcal{T}_1$-lower semicontinuous and $\mathcal{T}_2$-upper semicontinuous.

In a similar manner, the following result may be proved:

**Theorem 2.4.** *If $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise almost normal, then for every pair of real-valued functions f and g on X such that f is $\mathcal{T}_1$-lower semicontinuous, g is $\mathcal{T}_2$-upper semicontinuous with $g \leqq f$, and either g is $\mathcal{T}_1$-normal or f is $\mathcal{T}_2$-normal, there exists a $\mathcal{T}_1$-lower semicontinuous and $\mathcal{T}_2$-upper semicontinuous function h such that $g \leqq h \leqq f$.*

*Remark 2.2.* The results of theorems 1.4 and 1.5 remain valid if 'pairwise almost normal' be replaced by 'pairwise mildly normal'.

*Definition 2.4.* $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise Lindelöf* if every proper $\mathcal{T}_1$-closed set is $\mathcal{T}_2$-Lindelöf and every proper $\mathcal{T}_2$-closed set is $\mathcal{T}_1$-Lindelöf.

**Theorem 2.5.** *Every bi-second-axiom space $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise Lindelöf.*

Proof. Let $A_1$ be any proper, $\mathcal{T}_1$-closed subset of $X$. Let $\mathfrak{A} = \{\mathfrak{U}_\alpha : \alpha \in \Lambda\}$ be any $\mathcal{T}_2$-open covering of $A_1$. Let $\mathfrak{B}_2$ be a countable $\mathcal{T}_2$-open base for $\mathcal{T}_2$. Then each $\mathfrak{U}_\alpha$ is a union of members of $\mathfrak{B}_2$. It follows that there exists a subfamily $\mathfrak{C}_2$ of $\mathfrak{B}_2$ each member of which is contained in some member of $\mathfrak{A}$ and which covers $X$. Then the family of all members of $\mathfrak{A}$ which contain these members of $\mathfrak{C}_2$ is a countable subcovering of $\mathfrak{A}$ and hence $A_1$ is $\mathcal{T}_2$-Lindelöf. Similarly, every proper $\mathcal{T}_2$-closed set is $\mathcal{T}_1$-Lindelöf and thus $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise Lindelöf.

**Theorem 2.6.** *$(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise Lindelöf if and only if for each non-empty $U \in \mathcal{T}_2$, $(X, \mathcal{T}_1(U))$, where $\mathcal{T}_1(U) = \{X, \emptyset, U^* \cup U : U^* \in \mathcal{T}_1\}$ is Lindelöf and for each non-empty $V \in \mathcal{T}_1$, $(X, \mathcal{T}_2(V))$ where $\mathcal{T}_2(V) = = \{X, \emptyset, V^* \cup V : V^* \in \mathcal{T}_2\}$ is Lindelöf.*

Proof. Similar to the proof of theorem 3.3 in [9].

**Theorem 2.7.** *Every pairwise continuous image of a pairwise Lindelöf space is pairwise Lindelöf.*

Proof. Similar to the proof of theorem 3.9 in [9].

**Theorem 2.8.** *Every $\mathcal{T}_1$-closed and every $\mathcal{T}_2$-closed subset of a pairwise Lindelöf space is pairwise Lindelöf.*

Proof. Similar to the proof of theorem 3.7 in [9].

**Theorem 2.9.** *Every pairwise almost regular, pairwise Lindelöf space is pairwise mildly normal.*

Proof. Let $X$ be a pairwise almost regular, pairwise Lindelöf space. Let $A$ and $B$ be disjoint subsets of $X$ such that $A$ is (1, 2)-regularly closed and $B$ is (2, 1)-regularly closed. Then for each $x \in A$, $x \in X \sim B$. Since $X$ is pairwise almost regular, therefore there exists a (2, 1)-regularly open set $U_x$ such that $x \in U_x \subseteqq \mathcal{T}_1\text{-cl } U_x \subseteqq \subseteqq X \sim B$. It follows that $\{U_x : x \in A\}$ is a $\mathcal{T}_2$-open cover of $A$ such that $\mathcal{T}_1\text{-cl } U_x \cap B = \emptyset$ for all $U_x$. Similarly, for each $y \in B$, $y \in X \sim A$ and therefore there exists a (1, 2)-regularly open set $V_y$ such that $(\mathcal{T}_2\text{-cl } V_y) \cap A = \emptyset$ and $\{V_y : y \in B\}$ is a $\mathcal{T}_1$-open cover of $B$. Since $X$ is pairwise Lindelöf, therefore $A$ is $\mathcal{T}_2$-Lindelöf and $B$ is $\mathcal{T}_1$-Lindelöf. It follows that there exist countable subcovers $\{U_n : n = 1, 2, \ldots\}$ of $\{U_x : x \in A\}$ and $\{V_n : n = 1, 2, \ldots\}$ of $\{V_y : y \in B\}$. For each $n$, let

$$U_n^* = U_n \sim \bigcup \{\mathcal{T}_2\text{-cl } V_p : p \leqq n\}$$

and $V_n^* = V_n \sim \cup \{\mathcal{T}_1\text{-cl } U_p : p \leqq n\}$. Since $U_n^* \cap V_m = \emptyset$ for all $m \leqq n$, therefore $U_n^* \cap V_m^* = \emptyset$ for all $m \leqq n$. Similarly, $U_n^* \cap V_m^* = \emptyset$ for all $n \leqq m$ and hence $U_n^* \cap V_m^* = \emptyset$ for all $m$, $n$. Let $U = \cup \{U_n^* : n = 1, 2, \ldots\}$ and $V = \cup \{V_n^* : n = 1, 2, \ldots\}$. Since $(\mathcal{T}_1\text{-cl } \mathfrak{U}_p) \cap B = \emptyset$ and $(\mathcal{T}_2\text{-cl } V_p) \cap A = \emptyset$ for all $p$, therefore $U$ and $V$ are disjoint sets such that $U$ is $\mathcal{T}_2$-open, $V$ is $\mathcal{T}_1$-open and $A \subseteq U$, $B \subseteq V$. Hence $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise mildly normal.

**Theorem 2.10.** *Every pairwise regular, pairwise Lindelöf space is pairwise normal.*

PROOF. Similar to the proof of theorem 2.9 above.

*Corollary 2.1.* [KELLY, 3]. Every bi-second-axiom pairwise regular space is pairwise normal.

PROOF. Follows easily in view of theorem 2.5 and theorem 2.10 above.

*Corollary 2.2.* Every pairwise regular, bi-second-axiom space is hereditarily pairwise normal.

PROOF. Follows easily in view of theorems 2.5 and 2.10 above and theorem 1.2 in [9].

## 3. Pairwise almost completely regular spaces

*Definition 3.1.* $(X, \mathcal{T}_1, \mathcal{T}_2)$ is said to be *pairwise almost completely regular.* If for every $(i, j)$-regularly closed set $F$, and a point $x \notin F$, there exists a function $f: X \to [0, 1]$ such that $f$ is $\mathcal{T}_i$-upper-semicontinuous, $\mathcal{T}_j$-lower semicontinuous and $f(x) = 0$, $f(F) = 1$, $i \neq j$, $i, j = 1, 2$.

Obviously, every pairwise completely regular space and every pairwise almost normal, bi-$T_1$ space is pairwise almost completely regular. Also, every pairwise almost completely regular space is pairwise almost regular.

**Theorem 3.1.** *Every pairwise almost regular, pairwise mildly normal space is pairwise almost completely regular.*

PROOF. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be pairwise almost regular and pairwise mildly normal. Let $A$ be an $(i, j)$-regularly closed subset of $X$ and let $x \notin A$. Since $X$ is pairwise almost regular, there exist disjoint sets $U$ and $V$ such that $A \subseteq U$, $x \in V$, $U$ is $\mathcal{T}_j$-open, $V$ is $\mathcal{T}_i$-open and $\mathcal{T}_i\text{-cl } U \cap \mathcal{T}_j\text{-cl } V = \emptyset$. Then, $\mathcal{T}_i$-cl $U$ and $\mathcal{T}_j$-cl $V$ are disjoint sets such that $\mathcal{T}_i$-cl $U$ is $(i, j)$-regularly closed and $\mathcal{T}_j$-cl $V$ is $(j, i)$-regularly closed. Since $X$ is pairwise mildly normal, therefore there exists, in view of Theorem 2.2, a function $f: X \to [0, 1]$ such that $f(\mathcal{T}_j\text{-cl } V) = \{0\}$ $f(\mathcal{T}_i\text{-cl } U) = \{1\}$. $f$ is $\mathcal{T}_i$-upper semicontinuous and $\mathcal{T}_j$-lower semicontinuous. Also, $f(x) = 0$ and $f(A) = \{1\}$. Therefore $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise almost completely regular.

Incidently, we prove the following

**Theorem 3.2.** *Every pairwise regular, pairwise mildly normal space is pairwise completely regular.*

PROOF. Let $X$ be pairwise regular and pairwise mildly normal. Let $A$ be a $\mathcal{T}_1$-closed subset of $X$ and let $x \notin A$. Since $X$ is pairwise regular, there exist disjoint sets $U$ and $V$ such that $x \in V$, $A \subseteq U$, $U$ is $\mathcal{T}_2$-open, $V$ is $\mathcal{T}_1$-open and $\mathcal{T}_1$-cl $U \cap \cap \mathcal{T}_2$-cl $V = \emptyset$. Now, $\mathcal{T}_1$-cl $U$ is a (1, 2)-regularly closed set. Since $X$ is pairwise mildly normal, therefore there exists a function $f: X \to [0, 1]$ such that $f(\mathcal{T}_2\text{-cl } V) = 0$, $f(\mathcal{T}_1\text{-cl } U) = \{1\}$, $f$ is $\mathcal{T}_1$-upper semicontinuous and $\mathcal{T}_2$-lower semicontinuous. Hence $X$ is pairwise completely regular.

*Corollary 3.1.* Every pairwise regular, pairwise normal space is pairwise completely regular.

*Definition 3.2.* [Lane, 4]. If $f$ is a real-valued function on $(X, \mathcal{T}_1, \mathcal{T}_2)$ which is $\mathcal{T}_1$-lower semicontinuous and $\mathcal{T}_2$-upper semicontinuous, then $\{x \in X: f(x) \leqq 0\}$ is said to be a *$\mathcal{T}_1$-zero set* with respect to $\mathcal{T}_2$ (or simply a $\mathcal{T}_1$-zero set) and $\{x \in X: 0 \leqq f(x)\}$ is said to be a *$\mathcal{T}_2$-zero set* with respect to $\mathcal{T}_1$ (or simply a $\mathcal{T}_2$-zero set).

Obviously, every $\mathcal{T}_1$-zero set is $\mathcal{T}_1$-closed and every $\mathcal{T}_2$-zero set is $\mathcal{T}_2$-closed. Also, for any real number $r$, $\{x \in X: r \leqq f(x)\}$ is a $\mathcal{T}_1$-zero set and $\{x \in X: f(x) \leqq r\}$ is a $\mathcal{T}_2$-zero set if $f$ is $\mathcal{T}_1$-upper semicontinuous and $\mathcal{T}_2$-lower semicontinuous. Also, every $\mathcal{T}_1$-zero set is of the form $\{x \in X: h(x) = 0\}$, where $h$ is $\mathcal{T}_1$-lower semicontinuous and $\mathcal{T}_2$-upper semicontinuous and $h \geqq 0$. Similarly, any $\mathcal{T}_2$-zero set is of the form $\{x \in X: h(x) = 0\}$ where $h$ is $\mathcal{T}_1$-upper semicontinuous, $\mathcal{T}_2$-lower semicontinuous and $h \geqq 0$.

**Theorem 3.3.** *In a pairwise almost completely regular space every $(j, i)$-regularly open set containing a point $x$ contains a $\mathcal{T}_i$-zero set which is a $\mathcal{T}_j$-neighbourhood of $x$.*

Proof. Let $(X, \mathcal{T}_1, \mathcal{T}_2)$ be pairwise almost completely regular. Let $U$ be a $(j, i)$-regularly open set containing a point $x \in X$. Then $x \in X \sim U$ and $X \sim U$ is a $(j, i)$-regularly closed set. Therefore there exists a function $f: X \to [0, 1]$ such that $f$ is $\mathcal{T}_j$-upper semicontinuous, $\mathcal{T}_i$-lower semicontinuous such that $f(x) = 0$, $f(X \sim U) = \{1\}$. Let $V = \{x: f(x) \leqq 1/3\}$ and let $W = \{x: f(x) < 1/3\}$. Since $f$ is $\mathcal{T}_j$-upper semicontinuous and $\mathcal{T}_i$-lower semicontinuous, therefore $W$ is $\mathcal{T}_j$-open and $V$ is a $\mathcal{T}_i$-zero set. Also, $x \in W \subseteq V \subseteq X \sim (X \sim U) = U$. It follows that $V$ is a $\mathcal{T}_i$-zero set which is a $\mathcal{T}_j$-neighbourhood of $x$. Hence the reresult.

**Theorem 3.4.** *For a space $(X, \mathcal{T}_1, \mathcal{T}_2)$ the following are equivalent:*

(a) *$X$ is pairwise almost completely regular.*

(b) *Every $(i, j)$-regularly closed set $A$ is expressible as an intersection of some $\mathcal{T}_i$-zero sets which are $\mathcal{T}_j$-neighbourhoods of $A$.*

(c) *Every $(i, j)$-regularly closed set is identical with the intersection of all $\mathcal{T}_i$-zero sets which are $\mathcal{T}_j$-neighbourhoods of $A$.*

(d) *Every $(i, j)$-regularly open subset of $X$ containing a point contains a $\mathcal{T}_i$-cozero set containing that point.*

Proof. *(a)⇒(b).* If $A$ is an $(i, j)$-regularly closed set, then for each $x \notin A$, there exists a function $f_x: X \to [0, 1]$ such that $f_x(A) = 0$, $f_x(x) = 1$ and $f_x$ is $\mathcal{T}_i$-upper semicontinuous, $\mathcal{T}_j$-lower semicontinuous. Let $M_x = \{x: f_x(x) \leqq 1/3\}$ and let

$$N_x = \{x: f_x(x) \leqq 1/3\}.$$

Then, $A \subseteq \{x: f_x(x) < 1/3\} \subseteq \{x: f_x(x) \leq 1/3\}$, that is, $A \subseteq N_x \subseteq M_x$ where $N_x$ is $\mathscr{T}_j$-open and $M_x$ is a $\mathscr{T}_i$-zero set. It can be easily verified now that $A = \bigcap_{x \notin A} M_x$.

*(b)⇒(c).* Obvious.

*(c)⇒(d).* Let $U$ be an $(i, j)$-regularly open subset of $X$ containing a point $x$. Then $X \sim U = \bigcap_{\alpha \in \Lambda} A_\alpha$ where each $A_\alpha$ is a $\mathscr{T}_i$-zero set which is a $\mathscr{T}_j$-neighbourhood of $x$. Since $x \notin X \sim U$, therefore $x \notin A_\alpha$ for some $\alpha \in \Lambda$. Thus $x \in X \sim A_\alpha \subseteq U$. Since $A_\alpha$ is a $\mathscr{T}_i$-zero set, therefore $X \sim A_\alpha$ is a $\mathscr{T}_i$-cozero set. Hence the result.

*(d)⇒(a).* Let $A$ be an $(i, j)$-regularly closed set $A$ and let $x \notin A$. Then $x \in X \sim A$ and $X \sim A$ is an $(i, j)$-regularly open set containing $x$. By hypothesis, there exists a $\mathscr{T}_i$-zero set $U$ such that $x \in U \subseteq X \sim A$. This means that there exists a function $f$ on $X$ which is $\mathscr{T}_i$-lower semicontinuous, $\mathscr{T}_j$-upper-semicontinuous and such that $f(x) \neq 0$. Define a function $g$ on $X$ such that $g(y) = 1 - f(y)/f(x)$ for each $y \in X$. Then $g$ is $\mathscr{T}_i$-upper semicontinuous, $\mathscr{T}_j$-lower semicontinuous such that $g(x) = 0$, $g(A) = \{1\}$. Hence $X$ is pairwise almost completely regular.

**Theorem 3.5.** *In a pairwise almost completely regular space, a countable set $A$ disjoint from an $(i, j)$-regularly closed set $F$ is disjoint from some $\mathscr{T}_i$-zero set containing $F$.*

Proof. Let $A = \{x_n : n = 1, 2, \ldots\}$. Since $A \cap F = \emptyset$, therefore for each $n = 1, 2, \ldots$ $\ldots$, $X \sim F$ is an $(i, j)$-regularly closed set containing $x_n$. Therefore, for each $n = 1, 2, \ldots$ there exists a $\mathscr{T}_i$-cozero-set $U_n$ such that $x_n \in U_n \subseteq X \sim F$. Then $F \subseteq \bigcap_{n=1}^{\infty} (X \sim U_n) = U$. It follows that $U$ is a $\mathscr{T}_i$-zero set containing $F$ which is disjoint with $A$.

**Theorem 3.6.** *Every pairwise dense subspace of a bi-open subspace of a pairwise almost completely regular space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise almost completely regular.*

Proof. Let $(Y, \mathscr{T}_{1y}, \mathscr{T}_{2y})$ be any subspace of $(X, \mathscr{T}_1, \mathscr{T}_2)$. If $Y$ is pairwise dense or $Y$ is bi-open, it can be proved as in theorems 2 and 3 in [7] that for any subset $A$ of $Y$, we have, $\mathscr{T}_{iy}$-int $\mathscr{T}_{jy}$-cl $A = (\mathscr{T}_i\text{-int } \mathscr{T}_j\text{-cl } A) \cap Y$. Thus, if $A$ be any relatively $(i, j)$-regularly open subset of $Y$ containing a point $y$ of $Y$, then $\mathscr{T}_i$-int $\mathscr{T}_j$-cl $A$ is an $(i, j)$-regularly open subset of $X$ containing $y$. Since $X$ is pairwise almost completely regular therefore there exists a function $f: X \to [0, 1]$ such that $f$ is $\mathscr{T}_i$-upper semicontinuous and $\mathscr{T}_j$-lower semicontinuous such that $f(\mathscr{T}_i\text{-int } \mathscr{T}_j\text{-cl } A) = \{1\}$ and $f(y) = 0$. Then $g = f/Y$ is a $\mathscr{T}_{iy}$-upper semicontinuous and $\mathscr{T}_{jy}$-lower semicontinuous function on $Y$ such that $g(y) = 0$ and $g(A) = \{1\}$. Hence $Y$ is pairwise almost completely regular.

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