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Radial extension of monotone Riemannian metrics on density matrices

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Introduction

Let $\mathcal{M}_n$ be the space of all positive definite $n \times n$ complex matrices of trace 1. A Riemannian metric k on $\mathcal{M}_n$ is said to be monotone if for every stochastic map T the following holds:

$$k_D(T(X), T(Y)) \leq k_D(X, Y) \qquad D \in \mathcal{M}_n X, Y \in T_D\mathcal{M}_n$$

(Recall that a linear map T is stochastic if it is completely positive and $T(\mathcal{M}_n) \subset \mathcal{M}_n$)

On the base of MOROZOVA and CHENTSOV's work [7] D. PETZ showed in [8] that for every monotone metric k there exists a symmetric and positive operator monotone function f such that

$$k_D(X, Y) = \mathrm{Tr}(\boldsymbol{K}_D(X)Y), \qquad \boldsymbol{K}_D^{-1} = f(\boldsymbol{L}_D \boldsymbol{R}_D^{-1})\boldsymbol{R}_D \tag{1}$$

where $\boldsymbol{L}_D, \boldsymbol{R}_D$ are the operators of left and right multiplication by D. A function f is positive operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for every positive matrices A, B with order r and for each integer r. Such an f is symmetric if $f(x) = xf(x^{-1})$, this condition implies that $\boldsymbol{K}_D(X^*) = \boldsymbol{K}_D(X)^*$, so $\boldsymbol{K}_D$ maps the space of selfadjoint operators into itself. In addition $f(x)$ possesses the following integral representation:

$$f(x) = \mu(\{0\})\frac{x+1}{2} + \int\limits_{(0,1]} \frac{1+t}{2}\left(\frac{x}{x+t} + \frac{x}{xt+1}\right) d\mu(t), \quad \mathrm{Re}(x) > 0 \tag{2}$$

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where μ is a probability measure on [0,1] (see [6]).

Let $\mathcal{M}_n^\circ$ be the space of all positive semidefinite $n \times n$ complex matrices of trace 1 and let $\mathcal{P}$ be the space of all one dimensional projection operators. $\mathcal{M}_n^\circ$ can be considered as the space of states of a phisical system and $\mathcal{P}$ as the space of pure states. A. Uhlmann studied in [10] the Bures-metric which is related to the generalization of the Berry phase to mixed states and he gave a geometric representation of this metric. From this representation follows that the Bures-metric admits an extension to $\mathcal{P}$ and this extension is proportional to the canonical Riemannian metric of $\mathcal{P}$. Note that the operator monotone function $\frac{x+1}{2}$ represents the Bures-metric.

On the other hand, D. Petz and other authors in [2] and [9] studied the geometry of the Kubo-Mori metric (or Bogolubov inner product) which is induced by the relative entropy functional. It is defined by the formula

$$\langle\!\langle A, B\rangle\!\rangle_D = \int_0^\infty Tr\,(D+t)^{-1}A(D+t)^{-1}B\,dt, \quad D \in \mathcal{M}_n,\ A, B \in T_D\mathcal{M}_n.$$

and the corresponding operator monotone function is $\frac{x-1}{\log x}$. From their results follows that the Kubo–Mori metric does not admit any extension to $\mathcal{P}$.

The purpose of this paper is to define a type of extension of a monotone Riemannian metric to $\mathcal{P}$ and to give a necessary and sufficient condition for the existence of this extension by the help of Petz's characterization given by (1).

Canonical metric on $\mathcal{P}$

In this section we will recall various definitions of Riemannian metrics on spaces that are diffeomorphic to $\mathcal{P}$.

First of all, $\mathcal{P}$ itself admits a Riemannian metric, namely the restriction of the Hilbert-Schmidt metric to $\mathcal{P}$. This metric has the following form:

$$g_P(X, Y) = \mathrm{Tr}(XY^*), \tag{3}$$

where $P \in \mathcal{P}$ and $X, Y \in T_P\mathcal{P}$.

$\mathcal{P}$ is diffeomorphic to the complex projective space $\boldsymbol{P}^{n-1}\mathbb{C}$ (complex one dimensional subspaces of $\mathbb{C}^n$). If $p \in \boldsymbol{P}^{n-1}\mathbb{C}$ and $z \in p$, $z = (z_1 \dots z_n)$, $|z| = 1$ then this identification is given by the following map:

$$p \mapsto \begin{pmatrix} z_1\bar{z}_1 & \dots & z_1\bar{z}_n \\ \vdots & \ddots & \vdots \\ z_n\bar{z}_1 & \dots & z_n\bar{z}_n \end{pmatrix} \tag{4}$$

Let P_0 be the projection to the complex line generated by $e_1 = (1, 0, \dots, 0) \in \mathbb{C}^n$. Using (4), $T_{P_0}\mathcal{P}$ can be identified with matrices of the following form:

$$\boldsymbol{v} = \begin{pmatrix} 0 & \bar{v}_2 & \dots & \bar{v}_n \\ v_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_n & 0 & \dots & 0 \end{pmatrix} \tag{5}$$

where $v_i \in \mathbb{C}$ for $i = 2, \dots, n$. Let $(\ ,\ )$ and $\langle\ ,\ \rangle = \mathrm{Re}(,)$ denote the standard Hermitian inner product and the corresponding real one on $\mathbb{C}^n$, respectively. Let S^{2n-1} be the unit sphere in $\mathbb{C}^n$ with respect to $\langle\ ,\ \rangle$ and let S^1 be the group of complex unit vectors acting on S^{2n-1} by left multiplication. The identification of $\boldsymbol{P}^{n-1}\mathbb{C}$ and the quotient S^{2n-1}/S^1 gives a convenient definition of the canonical Riemannian metric on $\boldsymbol{P}^{n-1}\mathbb{C}$ as follows.

Let r, r_* be the projection from S^{2n-1} to $\boldsymbol{P}^{n-1}\mathbb{C} \sim S^{2n-1}/S^1$ and its tangent map respectively. Let $z \in S^{2n-1}$ and $T_z S^{2n-1}$ be the tangent space at z and $T_{r(z)}\mathcal{P}^{n-1}\mathbb{C}$ at $r(z)$. Then the kernel of the linear map $r_{*,z}$ is the real line generated by iz. Let $V_z \subset T_z S^{2n-1}$ be the orthogonal complement of $\mathrm{Ker}\, r_{*,z}$ with respect to the restriction of $\langle\ ,\ \rangle$ to $T_z S^{2n-1}$. $r_{*,z}$ gives a linear isomorphism of V_z to $T_z\boldsymbol{P}^{n-1}\mathbb{C}$ and the restriction of $\langle\ ,\ \rangle$ to V_z can be projected to an inner product h_z on $T_z\boldsymbol{P}^{n-1}\mathbb{C}$ such that $r_{*,z}$ becomes an isometry. Since S^1 is a group of isometries of S^{2n-1}, h_z is actually a Riemannian metric and r a Riemannian submersion (see [1] II.2.29). Using the map defined by (4), h induces a Riemannian metric on $\mathcal{P}$, which simply will be denoted by h. An easy computation shows that $g = 2h$.

Another definition comes from the isomorphism of $\boldsymbol{P}^{n-1}\mathbb{C}$ to the homogeneous space $U(n)/U(1)\times U(n-1)$ ($U(k)$ is the space of $k \times k$ unitary matrices). Let $E \in \boldsymbol{Q} = U(n)/U(1)\times U(n-1)$ be the left coset corresponding to $U(1)\times U(n-1)$ and T_E be the tangent space at E. The left action of an element $U \in U(1)\times U(n-1)$ on $\boldsymbol{Q}$ fixes E so its tangent map $U_{*,E}$

at E maps T_E onto itself. The homomorphism $U \to U_{*,E} \in GL(T_E)$ is called the isotropy representation of $\boldsymbol{Q}$. Since $U(1) \times U(n-1)$ is compact, one can use the Haar measure to define an inner product on T_E such that the elements of the isotropy representation are isometries. Since $U(n)$ acts transitively on $\boldsymbol{Q}$, this inner product induces a left invariant Riemannian metric. Moreover, the isotropy representation of $\boldsymbol{P}^{n-1}\mathbb{C}$ is irreducible (it has no nontrivial invariant subspaces) so this metric is unique up to a scalar factor (see [1] II.2.43).

In the last definition we will consider $\boldsymbol{P}^{n-1}\mathbb{C}$ as a complex $n-1$ dimensional manifold, isomorphic to $\mathbb{C}^n - \{0\}/\mathbb{C}^*$ where $\mathbb{C}^*$ is the group of nonzero complex numbers acting on $\mathbb{C}^n - \{0\}$ by left multiplication. Let us define the following function and the corresponding 2-form on $\mathbb{C}^n - \{0\}$:

$$f(z) = \ln(z^1\bar{z}^1 + z^2\bar{z}^2 + \cdots + z^n\bar{z}^n)$$

$$\tilde{\Phi} = -4i\partial\bar{\partial}f = -4i\sum_{i,j}\frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} dz^i \wedge d\bar{z}^j$$

It can be shown that this form is constant on complex lines so it can be projected to a 2-form Φ on $\mathbb{C}^n - \{0\}/\mathbb{C}^*$. If we set $g(X,Y) = \Phi(JX,Y)$ where J is the complex structure on $\mathbb{C}^n - \{0\}/\mathbb{C}^*$ we get the so called *Fubini-Study* metric (see [5] IX.6.3).

Definition of the radial extension

Let $\mathcal{M}_n' \subset \mathcal{M}_n$ be the set of the non degenerate elements of $\mathcal{M}_n$, i.e. the set of matrices whose eigenvalues are all distinct. This space is open and dense in $\mathcal{M}_n$. On $\mathcal{M}_n'$ we will consider the affine coordinate system, it consists of only one coordinate chart (ϕ, U) where $\phi\colon \mathcal{M}_n' \to \boldsymbol{R}^{n^2-1}$, $\phi(D) = D - I/n$ and $U = \phi(\mathcal{M}_n')$ is open in $\boldsymbol{R}^{n^2-1}$. The tangent space $T_D\mathcal{M}_n'$ at D is the space of traceless self-adjoint matrices.

Let us define a projection $\pi : \mathcal{M}_n' \to \mathcal{P}$ as follows. Let $\pi(D)$ be the projection to the one-dimensional eigenspace corresponding to the largest eigenvalue of D. This map is smooth (see [3] II.5.8), moreover $\mathcal{M}_n'$ is a smooth fibre bundle over $\mathcal{P}$ with projection π (see [4], I.5). The fibre space can be taken $\pi^{-1}(P_{\bar{\boldsymbol{e}}_1})$ where $\bar{\boldsymbol{e}}_1$ is the line generated by $\boldsymbol{e}_1 = (1, 0, \ldots, 0)^T \in \mathbb{C}^n$ and $P_{\bar{\boldsymbol{e}}_1}$ is the projection to $\bar{\boldsymbol{e}}_1$.

Let $\pi_{*,D}$ be the tangent map of π at D and let H_D be the orthogonal complement of $\operatorname{Ker}\pi_{*,D}$ in $T_D\mathcal{M}_n'$ with respect to a fixed monotone Riemannian metric k_D. Since $\pi_{*,D}$ is surjective, the restriction of $\pi_{*,D}$ gives a linear isomorphism between H_D and $T_{\pi(D)}\mathcal{P}$. If $\boldsymbol{v} \in T_{\pi(D)}\mathcal{P}$ then we can define a unique lift $\widehat{\boldsymbol{v}} \in H_D$ of $\boldsymbol{v}$ such that $\pi_{*,D}(\widehat{\boldsymbol{v}}) = \boldsymbol{v}$. Using this lift we can define the following inner product $g^D_{\pi(D)}$ on $T_{\pi(D)}\mathcal{P}$:

$$g^D_{\pi(D)}(\boldsymbol{u},\boldsymbol{v}) = k_D(\widehat{\boldsymbol{u}},\widehat{\boldsymbol{v}}) \qquad \boldsymbol{u},\boldsymbol{v} \in T_{\pi(D)}\mathcal{P}$$

We will say that a sequence $D_m \in \mathcal{M}'_n$ is *radial* at $P \in \mathcal{P}$ if $\pi(D_n) = P$ for every $m \in \boldsymbol{N}$ and D_m is convergent to P as m goes to infinity. Now we can define the radial extension of k.

Definition. We say that a smooth metric g on $\mathcal{P}$ is the radial extension of k if for every $p \in \mathcal{P}$, $\boldsymbol{u},\boldsymbol{v} \in T_p\mathcal{P}$ and for every radial sequence D_n at p

$$\lim_{m\to\infty} g^{D_m}_p(\boldsymbol{u},\boldsymbol{v}) = g_p(\boldsymbol{u},\boldsymbol{v}).$$

In the next section we give a necessary and sufficient condition for the existence of the radial extension.

The main existence theorem

Theorem. *Let k be a monotone Riemannian metric on $\mathcal{M}_n$ and let $f : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ be the corresponding operator monotone function. The radial extension g of k exists if and only if $f(0) \neq 0$. In this case $g = \frac{1}{f(0)}h$ where h is the canonical Riemannian metric on $\mathcal{P}$ defined by* (3).

Proof. For any unitary matrix U and $D \in \mathcal{M}'_n$ we have:

$$\pi(UDU^{-1}) = U\pi(D)U^{-1}.$$

Differentiation of this equality gives

$$\pi_{*,UDU^{-1}}(UXU^{-1}) = U\pi_{*,D}(X)U^{-1}, \qquad X \in T_D\mathcal{M}'_n.$$

Since k is unitary invariant and the action $U.U^{-1}$ is invertible, $U(\operatorname{Ker}\pi_{*,D})U^{-1} = \operatorname{Ker}\pi_{*,UDU^{-1}}$ and $UH_DU^{-1} = H_{UDU^{-1}}$. Moreover, for any $\boldsymbol{v} \in T_{\pi(D)}\mathcal{P}$, $U\widehat{\boldsymbol{v}}U^{-1} = \widehat{U\boldsymbol{v}U^{-1}}$; hence we get

$$g^D_{\pi(D)}(\boldsymbol{u},\boldsymbol{v}) = g^{UDU^{-1}}_{U\pi(D)U^{-1}}(U\boldsymbol{u}U^{-1}, U\boldsymbol{v}U^{-1}). \tag{6}$$

From this equality follows that it is sufficient to compute g^D if only D is diagonal and $\pi(D) = P_{\bar{\boldsymbol{e}}_1}$. Let us suppose D is diagonal and $\pi(D) = P_{\bar{\boldsymbol{e}}_1} = P_0$. For $X \in T_D\mathcal{M}'_n$ let $\lambda(t)$ be the largest eigenvalue of $D + tX$, $t \in \boldsymbol{R}$ and let $\boldsymbol{e}(t)$ be the unit eigenvector corresponding to $\lambda(t)$. For sufficiently small t, $D+tX \in \mathcal{M}'_n$ and $\lambda(t)$ and $\boldsymbol{e}(t)$ are smooth functions of t. Setting $T(t) = D + tX$ we have:

$$(T(t) - \lambda(t))\boldsymbol{e}(t) = 0.$$

Differentiating this expression we obtain that $\lambda'(0) = x_{11}$ and

$$\boldsymbol{e}'(0) = \left(0, \frac{x_{21}}{\lambda_1 - \lambda_2}, \dots, \frac{x_{n1}}{\lambda_1 - \lambda_n}\right)^T$$

(7)
$$\pi_{*,D}(X) = \begin{pmatrix} 0 & \frac{x_{12}}{\lambda_1-\lambda_2} & \dots & \frac{x_{1n}}{\lambda_1-\lambda_n} \\ \frac{x_{21}}{\lambda_1-\lambda_2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_{n1}}{\lambda_1-\lambda_n} & 0 & \dots & 0 \end{pmatrix}$$

where $\lambda_1, \dots \lambda_n$ are the eigenvalues of D, $\lambda_1 = \lambda(0)$ and $X = (x_{ij})$. If $X \in \operatorname{Ker} \pi_{*,D}$ then the expression of $\pi_{*,D}(X)$ gives:

$$X = \begin{pmatrix} x_{11} & 0 & \dots & 0 \\ 0 & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_{n2} & \dots & x_{nn} \end{pmatrix}.$$

Let $K_D^{-1} = f(L_D R_D^{-1}) R_D$ as in the Introduction. Since D is diagonal,

$$K_D(X) = \left(\frac{x_{ij}}{f\left(\frac{\lambda_i}{\lambda_j}\right)\lambda_j}\right);$$

hence we get $K_D(\operatorname{Ker} \pi_{*,D}) = \operatorname{Ker} \pi_{*,D}$. If $V \in H_D$ then the last equation gives

(8)
$$V = \begin{pmatrix} 0 & \bar{v}_2 & \dots & \bar{v}_n \\ v_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_n & 0 & \dots & 0 \end{pmatrix},$$

where $v_i \in \mathbb{C}$ for $i = 2, \dots n$. If $\boldsymbol{v} \in T_{P_0}\mathcal{P}$ then (5),(7) and (8) give

$$\widehat{\boldsymbol{v}} = \begin{pmatrix} 0 & (\lambda_1 - \lambda_2)\bar{v}_2 & \dots & (\lambda_1 - \lambda_n)\bar{v}_n \\ (\lambda_1 - \lambda_2)v_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ (\lambda_1 - \lambda_n)v_n & 0 & \dots & 0 \end{pmatrix}.$$

Now we can express g^D:

$$g^D(\boldsymbol{u},\boldsymbol{v}) = 2\,\mathrm{Re}\sum_{i=2}^{n} \frac{(\lambda_1-\lambda_i)^2}{f(\frac{\lambda_i}{\lambda_1})\lambda_1} u_i\bar{v}_i \tag{9}$$

where $\boldsymbol{u},\boldsymbol{v} \in T_{P_0}\mathcal{P}$.

Let us consider now the general case. Let D_m be a radial sequence at P and let $\boldsymbol{u},\boldsymbol{v} \in T_P\mathcal{P}$. Let B_P^m be linear operators on $T_P\mathcal{P}$ such that

$$g_P^{D_m}(\boldsymbol{u},\boldsymbol{v}) = h_P(B_P^m\boldsymbol{u},\boldsymbol{v}).$$

Let U_m be unitary operators such that $D_m^o = U_m D_m U_m^{-1}$ is diagonal and $\pi(D_m^o) = P_0$ where $P_0 = \bar{\boldsymbol{e}}_1$. Using (6) we have:

$$B_P^m = U_m^{-1} \circ B_{P_0}^m \circ U_m. \tag{10}$$

Since $\lim_{m\to\infty}\lambda_1^m = 1$ and $\lim_{m\to\infty}\lambda_i^m = 0$ for $i = 2\ldots n$, by (9)

$$\lim_{m\to\infty}\left\| B_{P_0}^m - \frac{1}{f(0)} I_{P_0}\right\|_{P_0} = 0 \tag{11}$$

where I_{P_0} is the identity map on $T_{P_0}\mathcal{P}$ and $\|\cdot\|_{P_0}$ is the operator norm induced by $\langle\ ,\ \rangle_{P_0}$. It follows from (10) that

$$\begin{aligned}
\left\| B_P^m - \frac{1}{f(0)} I_P\right\|_P &= \left\| U_m^{-1}\circ B_{P_0}^m \circ U_m - \frac{1}{f(0)} U_m^{-1}\circ I_{P_0}\circ U_m\right\|_P \\
&= \left\| U_m^{-1}\circ\left(B_{P_0}^m - \frac{1}{f(0)} I_{P_0}\right)\circ U_m\right\|_P \\
&\le \left\|U_m^{-1}\right\|_{P,P_0}\cdot\left\| B_{P_0}^m - \frac{1}{f(0)} I_{P_0}\right\|_{P_0}\cdot\left\|U_m\right\|_{P_0,P}.
\end{aligned}$$

Since U_m are isometries from $T_P\mathcal{P}$ to $T_{P_0}\mathcal{P}$, $\|U_m\|_{P,P_0} = 1$ and by (11) we obtain

$$\lim_{m\to\infty}\left\| B_P^m - \frac{1}{f(0)} I_P\right\|_P = 0. \qquad \square$$

Remark. In virute of formula (2) the condition $f(0) \neq 0$ is equivalent to $\mu(\{0\}) \neq 0$. $\mathcal{P}$ is a proper subset of the topological boundary of $\mathcal{M}_n$ for $n \geq 3$, so one can ask for the extension of a monotone metric to other points of the boundary. Since this boundary does not admit any differentiable manifold structure, it should be well-specified how the extension is understood. A detailed study of the extension of a monotone metric to the whole boundary with the aid of the generalization of the radial extension will be presented in a forthcoming paper.

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References

[1] S. GALLOT, D. HULIN and J. LAFONTAINE, Riemannian geometry, *Springer-Verlag, Berlin – Heidelberg – New York Inc.*, 1990.
[2] F. HIAI, D. PETZ and G. TOTH, Curvature in the geometry of canonical correlation, *J. Math. Phys.* (*to appear*).
[3] T. KATO, Perturbation theory for linear operators, *Springer-Verlag, Berlin – Heidelberg – New York Inc.*, 1980.
[4] S. KOBAYASHI and K. NOMIZU, Foundations of differential geometry, vol. I, *Wiley Intersience, New York – London*, 1963.
[5] S. KOBAYASHI and K. NOMIZU, Foundations of differential geometry, vol. II, *Wiley Intersience, New York – London – Sydney*, 1969.
[6] F. KUBO and T. ANDO, Means of positive linear operators, *Math. Ann.* **246** (1980), 205–224.
[7] E. MOROZOVA and N. CHENTSOV, Markov invariant geometry on state manifolds, *Itogi Nauki i Tekhniki* **36** (1990), 69–102 (in Russian).
[8] D. PETZ, Monotone metrics on matrix spaces, *Linear Algebra Appl.* (*to appear*).
[9] D. PETZ and G. TOTH, The bogoliubov inner product in quantum statistics, *Lett. Math. Phys.* **27** (1993), 205–216.
[10] A. UHLMANN, Density operators as an arena for differential geometry, *Rep. Math. Phys.* **33**, 253–263.

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