Semigroups with idempotent ideals

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To Professor A. Rapcsák on his 60th birthday

1. By a *semigroup with idempotent ideals* we mean a semigroup in which every ideal is idempotent. The class of commutative semigroups with idempotent ideals was characterized by S. LAJOS ([3], Corollary) as follows: a commutative semigroup is regular if and only if it is a semigroup with idempotent ideals. In case of omitting the condition of commutativity the "only if" part of the theorem still remains valid (see Proposition 1 below).

In Section 2 we get to a characterisation of semigroups with idempotent ideals through two preparatory theorems and we obtain S. Lajos's theorem mentioned above as a corollary. In Section 3 we give some information about the place of this class of semigroups among other classes of semigroups. (For a characterisation of a quite different kind of these semigroups see [2].)

For notions and concepts we refer to [1] with the only exception that the principal ideal generated by the element a will be denoted by (a).

2. First we give a condition that is necessary and sufficient for a principal ideal to be idempotent.

Theorem 1. *The principal ideal* (a) *of a semigroup* S *is idempotent if and only if* $a\in SaSaS$.

PROOF. Assume $a\in SaSaS$. Then

$$a\in(Sa)(SaS)\subseteqq(a)(a)=(a)^2,$$

whence $(a)\subseteqq(a)^2$. As the converse inclusion is always true, $(a)=(a)^2$.

Assume now $(a)=(a)^2$. Then

$$\begin{aligned}a\in(a)=(a)^5&=S^1aS^1\cdot S^1aS^1\cdot S^1aS^1\cdot S^1aS^1\cdot S^1aS^1=\\&=S^1aS^1S^1\cdot a\cdot S^1S^1aS^1S^1\cdot a\cdot S^1S^1aS^1\subseteqq\\&\subseteqq SaSaS,\end{aligned}$$

as asserted.

Theorem 2. *Let* S *be a semigroup and* I *an ideal of* S. *If* $(a)=(a)^2$ *for every element* a *of* I, *then* $I=I^2$, *too.*

PROOF. Assume $(a)=(a)^2$ for each $a\in I$. Then $a\in(a)^2\subseteqq I^2$ for each $a\in I$. Hence $I\subseteqq I^2$. On the other hand, $I^2\subseteqq I$ for every ideal I.

Corollary (S. Lajos's theorem). *A commutative semigroup is regular if and only if it is a semigroup with idempotent ideals.*

PROOF. If S is a semigroup with idempotent ideals, then every element a of S can be represented in the form

$$a = xayaz = a(xyz)a \quad (x, y, z\in S)$$

by Theorem 1 and by commutatity. Thus, S is regular.

Assume, conversely, that S is regular and let a be an arbitrary element of S. Then there exists an $x\in S$ such that $axa=a$ and

$$a = (ax)a\in(a)(a) = (a)^2.$$

Hence $(a)=(a)^2$ and thus S is a semigroup with idempotent ideals by Theorem 2.

Remark. The statement converse to this theorem does not hold: $I^2=I$ does not imply $(a)^2=(a)$ for each element a of an ideal I (consider $I=S$ in Example 2 below).

Now we prove the main theorem of this note.

Theorem 3. *The following assertions are equivalent:*
(A) *S is a semigroup with idempotent ideals;*
(B) *S is a semigroup with idempotent principal ideals;*
(C) *$a\in SaSaS$ for every element a of the semigroup S.*

PROOF. (A) implies (B) trivially. (B) implies (C) by Theorem 1. (C) implies (A) as follows: if (C) is satisfied, then every principal ideal of S is idempotent by Theorem 1 and thus every ideal of S is idempotent, too, by Theorem 2.

3. In the second half of the proof of Corollary to Theorem 2 we have verified the following

Proposition 1. *Every ideal of a regular semigroup is idempotent.*

It can be shown by the same way that *every ideal of a left regular or right regular or intra-regular semigroup is idempotent.*

It is well-known that a semigroup in which every ideal is prime is a semigroup with idempotent ideals. (This is a simple consequence of Corollary 3.2. in [4].) As the matter of fact, *the class of semigroup with idempotent ideals is properly wider than the class of semigroups with prime ideals,* even inside the class of commutative semigroups. This assertion can be verified by the following example:

Example 1. Consider the commutative semigroup $S=\{0, a, b\}$ in which every element is idempotent and $ab=0$. Every ideal of S is idempotent, but the principal ideal (0) is not prime because $ab\in(0)$, $a\notin(0)$ and $b\notin(0)$.

An ideal I of a semigroup S is said to be *reproduced* by S if $SI=IS=I$ ([5], Einleitung). We show that *the class of semigroups reproducing their ideals is properly* *wider than the class of semigroups with idempotent ideals* (see Proposition 2 and Example 2).

Proposition 2. *Every ideal I of a semigroup S with idempotent ideals is reproduced by S.*

PROOF. In fact, $I=I^2\subseteqq SI\subseteqq I$ whence $SI=I$. Similary, $IS=I$.

Example 2. Let $S=\{0, 1, a\}$ be a commutative semigroup in which $a^2=0$. Every ideal of S is reproduced by S, but the principal ideal (a) is not idempotent: $(a)^2=$ $=(0)\neq(a)$. We remark that this semigroups is not regular because the equation $axa=a$ cannot be solved.

References

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